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θ_C from the dihedral flavor symmetries D_7 and D_{14}

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Abstract: In [1, 2] it has been mentioned that the Cabibbo angle θ_C might arise from a dihedral flavor symmetry (which is broken to different (directions of) subgroups in the up and the down quark sector). Here we construct a low energy model which incorporates this idea. The gauge group is the one of the Standard Model and $D_7 \times Z_2^{(aux)}$ serves as flavor symmetry. The additional $Z_2^{(aux)}$ is necessary in order to maintain two sets of Higgs fields, one which couples only to up quarks and another one coupling only to down quarks. We assume that D_7 is broken spontaneously at the electroweak scale by vacuum expectation values of $\mathrm{SU}(2)_L$ doublet Higgs fields. The quark masses and mixing parameters can be accommodated well. Furthermore, the potential of the Higgs fields is studied numerically in order to show that the required configuration of the vacuum expectation values can be achieved. We also comment on more minimalist models which explain the Cabibbo angle in terms of group theoretical quantities, while θ^q_{13} and θ^q_{23} vanish at leading order. Finally, we perform a detailed numerical study of the lepton mixing matrix V_{MNS} in which one of its elements is entirely determined by the group theory of a dihedral symmetry. Thereby, we show that nearly tri-bi-maximal mixing can also be produced by a dihedral flavor group with preserved subgroups.

Keywords: Discrete and Finite Symmetries, Neutrino Physics, Quark Masses and SM Parameters.

Contents

1. Introduction

Discrete groups have been widely used as flavor symmetry. However, only in some special cases there is a direct connection between the flavor group G_F and the resulting mixing pattern for the fermions, i.e. a correlation which does not rely on further parameter equalities not induced by G_F. This has probably been first exploited in the successful A_4 models [3] to predict tri-bi-maximal mixing (TBM). Later on this has been studied in a more general way with the help of so-called mass-independent textures [4]. In [2] the groups which can produce TBM (at least partly), if non-trivial subgroups are preserved in the neutrino and charged lepton sector, have been constructed, a recipe is given to generate other mixing patterns in the same way and a possibility to explain the Cabibbo angle θ_C has been briefly mentioned. In our recent paper [1] we conversely derived the possible mass matrix structures arising from dihedral symmetries, if they are broken in a non-trivial way and studied how they can lead to maximal atmospheric mixing and vanishing θ_{13} as well as to a prediction of θ_C. The key feature in all these studies is the existence of residual subgroups in different sectors of the theory. Especially, the fact that sizable mixing results from the mismatch of two different (directions of) subgroups is important. For example, in the group A_4 (T') [3, 5] which has been studied in great detail TBM in the lepton sector is predicted, if one assumes that the left-handed leptons transform as a triplet under A_4 (T'), and the left-handed conjugate leptons, e^c, μ^c and τ^c, as the three non-equivalent one-dimensional representations of the group. There exist two sets of gauge singlets which transform non-trivially under A_4 (T'): one set only couples to neutrinos at the leading order (LO), while the other one only to charged leptons (fermions). The first one breaks A_4 (T') spontaneously down to Z_2 (Z_4) and the latter one down to Z_3. The lepton mixing then stems from two sectors in which different subgroups of A_4 (T') are conserved. In contrast to this, the up quark and down quark mass matrix preserve the same subgroup at LO [5]. In [1, 2] it has been observed that θ_C or equivalently the CKM matrix element $|V_{us}|$ can be predicted with a dihedral flavor symmetry in terms of group theoretical indices only, such as the index n of the group D_n, the index j of the representation under which the (left-handed) quarks transform and the misalignment of the two different (directions of) subgroups $Z_2 =< \mathrm{B}\,\mathrm{A}^{m_u} >$ and $Z_2 =< \mathrm{B}\,\mathrm{A}^{m_d} >$:

$$|V_{us}| = \left|\cos\left(\frac{\pi\,(m_u - m_d)\,\mathrm{j}}{n}\right)\right| \tag{1.1}$$

There is a crucial difference between these two examples using a dihedral group and A_4 (T') as flavor symmetry, namely the issue whether the representations under which the Higgs (flavon) fields transform are chosen or not. In [1, 2] it was assumed that the transformation properties of the Higgs fields are not selected by hand, but it was only required that their vacuum expectation values (VEVs) conserve the relevant subgroup of the flavor symmetry. Due to this the resulting mass matrices are only determined by the choice of the fermion representations, the flavor group and the preserved subgroups, but not by the choice of the scalar fields. However, in the case of the A_4 (T') it is necessary to choose the transformation properties of the scalar fields properly, i.e. one has to exclude scalars which transform as non-trivial singlets under A_4 (T') and couple to neutrinos at LO, in order to arrive at the TBM scenario [2, 4 − 6].

In this paper we investigate the idea of [1, 2] by constructing a viable (low energy) model for the quark sector. The gauge group is chosen to be the one of the Standard Model (SM), while the smallest flavor symmetry which is appropriate is D_7. This group has already been employed as flavor symmetry in [7] in order to produce textures in the up and down quark mass matrices which lead to a prediction of $\sin(2\,\beta)$ $(\sin(2\,\phi_1))$, which is the CP violation parameter in B decays. In our analysis we study the mass matrices numerically in order to demonstrate that all quark masses and mixing parameters can be accommodated. We

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discuss the Higgs potential under the assumption that all involved fields are copies of the SM Higgs doublet. Furthermore, instead of accommodating all quark mixing angles at LO it is also worth studying setups in which the Cabibbo angle is predicted in terms of group theoretical quantities, while the two other mixing angles are zero. This can be done in at least two different ways which we will discuss. Finally, we motivate possible extensions of the model to the lepton sector by performing a detailed numerical study. Additionally, we show that nearly TBM can be also accommodated by using a dihedral flavor symmetry.

The paper is organized as follows: in section 2 we review the findings of [1] which we explore in more detail; section 3 treats the mixing matrix $V_{\rm CKM}$ only - in an analytic way as well as numerically; in section 4 we study a model for the quark sector which incorporates the idea presented in [1, 2] and show that it fits both quark mixings and masses; in section 5 the Higgs potential, belonging to one of the models of section 4, is discussed and a numerical analysis proves that the advocated VEV structure can be achieved. Section 6 is devoted to ansätze in which only the Cabibbo angle is generated at LO. In section 7 we perform a similar analysis, as for the quark mixing matrix $V_{\rm CKM}$ in section 3, for the lepton mixing matrix $V_{\rm MNS}$. Thereby, we assume that the neutrinos are Dirac particles as all the other fermions and are normally ordered. Finally, we summarize our results in section 8. Appendix A contains the possible forms of the mixing matrices $V_{\rm CKM}$ and $V_{\rm MNS}$. In appendix B the group theory of D_7 is presented. Further details of the study of the Higgs sector are relegated to appendix C.

2. Basics

In this section we repeat the findings of [1] concerning the possible structure of (Dirac) mass matrices with a non-vanishing determinant. They are of the form:

$$M_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} , \qquad M_2 = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix} \tag{2.1}$$

$$M_3 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & D & E \end{pmatrix} , \tag{2.2}$$

$$M_4 = \begin{pmatrix} 0 & A & B \\ C & D & E \\ -C\,{\rm e}^{-i\,\phi\,{\rm j}} & D\,{\rm e}^{-i\,\phi\,{\rm j}} & E\,{\rm e}^{-i\,\phi\,{\rm j}} \end{pmatrix} \text{ and } M_5 = \begin{pmatrix} A & C & C\,{\rm e}^{-i\,\phi\,{\rm k}} \\ B & D & E \\ B\,{\rm e}^{-i\,\phi\,{\rm j}} & E\,{\rm e}^{-i\,\phi\,({\rm j}-{\rm k})} & D\,{\rm e}^{-i\,\phi\,({\rm j}+{\rm k})} \end{pmatrix} \tag{2.3}$$

where A, B, C, D, E are complex numbers which are products of Yukawa couplings and VEVs, $\phi = \frac{2\pi}{n}\, m$ (n: index of the dihedral group, m: index of the breaking direction) and ${\rm j}, {\rm k}$ are indices of representations. Regarding M_4 notice that we presented in [1] the transpose of this matrix. However, a transposition in general only corresponds to the exchange of the transformation properties of the left-handed and left-handed conjugate fields under the flavor symmetry and therefore does not change the group theoretical part of the discussion. These matrices are determined up to permutations of columns and rows which correspond to permutations among the three generations of the fields. We work in the SM and with the assumption that all Higgs fields H in the model are copies of the SM one. Therefore the displayed mass matrices are those for down-type fermions, i.e. down quarks and charged leptons. The corresponding ones for up-type fermions, i.e. up quarks and (Dirac) neutrinos, require some changes due to the fact that only the conjugates of the Higgs fields, $\epsilon\, H^{\star}$, couple to up-type fermions and that we use complex matrices for the two-dimensional representations of D_n. According to the rules of [1] M_4 and M_5 are of the form

$$M_4 = \begin{pmatrix} 0 & A & B \\ C\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & D\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & E\,\mathrm{e}^{i\,\phi\,\mathrm{j}} \\ -C & D & E \end{pmatrix} \quad \text{and} \quad M_5 = \begin{pmatrix} A & C\,\mathrm{e}^{i\,\phi\,\mathrm{k}} & C \\ B\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & D\,\mathrm{e}^{i\,\phi\,(\mathrm{j+k})} & E\,\mathrm{e}^{i\,\phi\,(\mathrm{j-k})} \\ B & E & D \end{pmatrix} \tag{2.4}$$

Explicit examples are given in section 4. We concentrate on the last two forms, M_4 and M_5, since we want to accommodate all masses and mixing parameters at tree level (apart from section 6) and also would like to have the same mass matrix structure for up quarks (Dirac neutrinos) and down quarks (charged leptons).
Let us briefly mention the origin of the matrix structures M_4 and M_5. The flavor symmetry is a single-valued dihedral group D_n with arbitrary index n. The preserved subgroup is in both cases $Z_2 =< \mathrm{BA}^m >$ where $m = 0, 1, \ldots, n-1$. This subgroup allows non-vanishing VEVs for the following one-dimensional representations: $\underline{\mathbf{1}}_{\mathbf{1}}$ (is always allowed to have a VEV), $\underline{\mathbf{1}}_{\mathbf{3}}$ for m even and $\underline{\mathbf{1}}_{\mathbf{4}}$ for m odd. All two-dimensional representations acquire a so-called structured VEV, i.e. for two fields $\psi_{1,2}$ transforming as an irreducible two-dimensional representation $\underline{\mathbf{2}}_{\mathbf{p}}$ their VEVs have to have the correlation: $\langle\psi_1\rangle = \langle\psi_2\rangle \mathrm{e}^{-\frac{2\,\pi\,i\,p\,m}{n}}$. The notation of the representations used here is according to the one given in [1]. In case of M_4 we take the left-handed fields L to transform as $\underline{\mathbf{1}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{j}}$ under the dihedral group, and the left-handed conjugate fields L^c transform as the three singlets $\underline{\mathbf{1}}_{\mathrm{i}_1} + \underline{\mathbf{1}}_{\mathrm{i}_2} + \underline{\mathbf{1}}_{\mathrm{i}_3}$. A study of all possible assignments shows that one of the entries in the first row needs to be zero in order to prevent the determinant of the matrix from being zero. The matrix structure M_5 arises, if both left-handed and left-handed conjugate fermions transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$, $L \sim (\underline{\mathbf{1}}_{\mathbf{i}}, \underline{\mathbf{2}}_{\mathbf{j}})$ and $L^c \sim (\underline{\mathbf{1}}_{\mathbf{l}}, \underline{\mathbf{2}}_{\mathbf{k}})$. Here the constraint $\det(M) \neq 0$ enforces the (11) element of the mass matrix to be non-zero, i.e. $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{l}}$ has to have a non-vanishing VEV.
To study the mixing matrices arising from M_4 and M_5 for down-type as well as up-type fermions we observe that the products $M_i\, M_i^{\dagger}$, $i = 4, 5$, can be written in the general form

$$\begin{pmatrix} a & b\,\mathrm{e}^{i\,\beta} & b\,\mathrm{e}^{i\,(\beta+\phi\,\mathrm{j})} \\ b\,\mathrm{e}^{-i\,\beta} & c & d\,\mathrm{e}^{i\,\phi\,\mathrm{j}} \\ b\,\mathrm{e}^{-i\,(\beta+\phi\,\mathrm{j})} & d\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & c \end{pmatrix}$$

where a, b, c, d and β are real functions of A, B, C, D and E. The phase β lies in the interval $[0, 2\,\pi)$. Since we work in the basis in which the left-handed fields are on the left-hand side and the left-handed conjugate fields on the right-hand side, the unitary matrix which diagonalizes $M_i\, M_i^{\dagger}$ acts on the left-handed fields and therefore determines the physical mixing matrices. The three eigenvalues are given as $(c-d)$, $\frac{1}{2}\,(a+c+d-\sqrt{(a-c-d)^2+8\,b^2})$

and $\frac{1}{2}\,(a+c+d+\sqrt{(a-c-d)^2+8\,b^2})$. Assuming this ordering of the eigenvalues the mixing matrix U which fulfills $U^\dagger\,M_i\,M_i^\dagger\,U = \text{diag}$ is of the form:

$$U = \begin{pmatrix} 0 & \cos(\theta)\,\mathrm{e}^{i\,\beta} & \sin(\theta)\,\mathrm{e}^{i\,\beta} \\ -\frac{1}{\sqrt{2}}\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & \frac{\cos(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} \end{pmatrix}$$

The angle θ is determined to be $\tan(2\,\theta) = \frac{2\sqrt{2}\,b}{c+d-a}$. Therefore it lies in the interval $[0, \frac{\pi}{2})$. If the three eigenvalues are not degenerate, the eigenvectors are determined by them up to phases.[1] Therefore the variants of the mixing matrix U are given by permutations of the columns. With this at hand we can look for possible interesting structures in the mixing matrix which is just the product of two matrices of this form, i.e. $V = W_1^T\,W_2^\star$ with W_i being a variant of U. For $V = V_{\text{CKM}}$ we have $W_1 \equiv U_u$ which is the unitary matrix diagonalizing the up quark mass matrix and $W_2 \equiv U_d$ which is the corresponding matrix for the down quarks. In case of $V = V_{\text{MNS}}$, W_1 is equivalent to U_l and W_2 to U_ν.[2] The matrix W_i contains the group theoretical phase ϕ_i according to the breaking direction m_i, the angle θ_i and the phase β_i. For $W_1 \equiv U_u$ we also use the notation ϕ_u, m_u, θ_u and β_u. An analogous convention is used for U_d, U_l and U_ν. It turns out that one of the elements is determined by the index j of the representation $\underline{\mathbf{2}}_{\mathbf{j}}$ under which two of the left-handed fields transform and the difference of the group theoretical phases ϕ_1 and ϕ_2 only. The actual form of (the absolute value of) the element is

$$\frac{1}{2}\left|1+\mathrm{e}^{i\,(\phi_1-\phi_2)\,\mathrm{j}}\right| = \left|\cos\left((\phi_1-\phi_2)\,\frac{\mathrm{j}}{2}\right)\right| = \left|\cos\left(\frac{\pi}{n}\,(m_1-m_2)\,\mathrm{j}\right)\right| \tag{2.5}$$

Note that this value is only non-trivial, if $m_1 \neq m_2$, i.e. the (directions of the) subgroups which are preserved in the up quark (Dirac neutrino) sector and the down quark (charged lepton) sector are not the same. This element can be traced back to the eigenvectors which correspond to the eigenvalue $c-d$. Therefore the ordering of the eigenvectors in the up quark (Dirac neutrino) and down quark (charged lepton) sector determines in which position of the mixing matrix the fixed element appears.

In [1, 2] it was already mentioned that one can accommodate the CKM matrix element $|V_{us}|$ by $\cos(\frac{3\pi}{7}) \approx 0.2225$, i.e. by taking $n=7$ and for example $\mathrm{j}=3$, $m_u=1$ and $m_d=0$ in eq. (2.5). Here we show first which of the other elements of V_{CKM} can also be accommodated well by the form $\left|\cos(\frac{\pi}{n}\,(m_u-m_d)\,\mathrm{j})\right|$. The elements of V_{CKM} are precisely measured [8]

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.97383^{+0.00024}_{-0.00023} & 0.2272^{+0.0010}_{-0.0010} & (3.96^{+0.09}_{-0.09})\times 10^{-3} \\ 0.2271^{+0.0010}_{-0.0010} & 0.97296^{+0.00024}_{-0.00024} & (42.21^{+0.10}_{-0.80})\times 10^{-3} \\ (8.14^{+0.32}_{-0.64})\times 10^{-3} & (41.61^{+0.12}_{-0.78})\times 10^{-3} & 0.999100^{+0.000034}_{-0.000004} \end{pmatrix}$$

[1]Since the eigenvectors should be normalized their length is fixed to one.

[2]Throughout the paper we assume that the neutrinos are Dirac particles for simplicity. Therefore V_{MNS} has the same structure as V_{CKM}, i.e. there are no (additional) Majorana phases present in the lepton sector.

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together with the Jarlskog invariant [9] $J_{\rm CP} = (3.08^{+0.16}_{-0.18}) \times 10^{-5}$. We restrict ourselves to values of n smaller than 30, since then the group order is smaller than 60. Using eq. (2.5) we see that we can put the elements of the $1-2$ sub-block, i.e. $|V_{ud}|$, $|V_{us}|$, $|V_{cd}|$ and $|V_{cs}|$, into this form. As $|V_{cd}| \approx |V_{us}|$ holds to good accuracy, also $|V_{cd}|$ can be described well by $\cos(\frac{3\pi}{7})$. Furthermore $|V_{ud}| \approx |V_{cs}|$ can be approximated well as $\cos(\frac{\pi}{14}) \approx 0.9749$ which points towards the flavor group D_{14}. Note that the value of $|V_{ud}|$ as well as of $|V_{cs}|$ can be accommodated even a bit better with $\cos(\frac{2\pi}{27}) \approx 0.9730$. However, this needs the group D_{27} which is a group of order 54 and therefore already quite large. Note that, even if $|V_{us}|$ is taken to be $\cos(\frac{3\pi}{7})$, there is no unique solution which flavor symmetry has to be used and to which subgroup it has to be broken, since for example taking $\mathrm{j} = 1$, $m_u = 3$, $m_d = 0$ and $n = 7$ leads to $\left|\cos(\frac{\pi}{n}(m_u - m_d)\,\mathrm{j})\right| = |\cos(\frac{3\pi}{7})|$ as well as $\mathrm{j} = 3$, $m_u = 1$, $m_d = 0$ and $n = 7$. In the next section we study the cases $|V_{us}|$ and $|V_{cd}|$ equal to $\cos(\frac{3\pi}{7})$ and $|V_{ud}|$ and $|V_{cs}|$ equal to $\cos(\frac{\pi}{14})$ in greater detail and thereby check whether we can adjust the two other mixing angles θ^q_{13} and θ^q_{23} with the free angles θ_u and θ_d and also the Jarlskog invariant $J_{\rm CP}$ with the difference of the two phases β_u and β_d.

3. Analysis of $V_{\rm CKM}$ only

3.1 Remarks

There are six possible forms for U which correspond to different identifications of the eigenvalues. However, the fact that $m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$ allows only three of them, as the eigenvalue $\frac{1}{2}\,(a + c + d - \sqrt{(a-c-d)^2 + 8\,b^2})$ is smaller than $\frac{1}{2}\,(a + c + d + \sqrt{(a-c-d)^2 + 8\,b^2})$. Therefore, we will only vary the position of the eigenvector belonging to the eigenvalue $c - d$, while keeping the ordering of the two others fixed. The three different forms of the mixing matrix U are then:

$$U = \begin{pmatrix} 0 & \cos(\theta)\,\mathrm{e}^{i\,\beta} & \sin(\theta)\,\mathrm{e}^{i\,\beta} \\ -\frac{1}{\sqrt{2}}\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & \frac{\cos(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} \end{pmatrix}, \quad U' = \begin{pmatrix} \cos(\theta)\,\mathrm{e}^{i\,\beta} & 0 & \sin(\theta)\,\mathrm{e}^{i\,\beta} \\ -\frac{\sin(\theta)}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & \frac{\cos(\theta)}{\sqrt{2}} \\ -\frac{\sin(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & \frac{1}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} \end{pmatrix},$$

$$U'' = \begin{pmatrix} \cos(\theta)\,\mathrm{e}^{i\,\beta} & \sin(\theta)\,\mathrm{e}^{i\,\beta} & 0 \\ -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\,\mathrm{e}^{i\,\phi\,\mathrm{j}} \\ -\frac{\sin(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & \frac{\cos(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Combining them leads to nine distinct possibilities for the CKM matrix whose forms are displayed in appendix A. Since we already mentioned that we want to concentrate on the $1-2$ sub-block we only need to consider the four possible combinations which involve U and U'.

3.2 Numerical study

We now discuss the results of our fits to the CKM matrix. The forms of $V_{\rm mix}$ presented in appendix A show that two of the elements $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$ and $|V_{ts}|$ are determined by $\cos(\theta_{u,d})$ in each of the four different cases. As these elements are small, the free angles θ_u

and θ_d are restricted to be $\theta_{d,u} \approx \frac{\pi}{2}$. Therefore $\theta_{d,u}$ is expanded around $\frac{\pi}{2}$, $\theta_{d,u} = \frac{\pi}{2} - \epsilon_{d,u}$, $\epsilon_{d,u} > 0$. The resulting four CKM matrices are (up to the first order in $\epsilon_{u,d}$)

$$|V_{\rm CKM}^{11}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{3\pi}{7})\,\epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2}\,|(1+{\rm e}^{\frac{\pi}{7}\,i})\,\epsilon_d - 2\,{\rm e}^{i\,\alpha}\,\epsilon_u| \\ \cos(\frac{3\pi}{7})\,\epsilon_u & \frac{1}{2}\,|(1+{\rm e}^{\frac{\pi}{7}\,i})\,\epsilon_u - 2\,{\rm e}^{i\,\alpha}\,\epsilon_d| & 1 \end{pmatrix} \tag{3.1}$$

$$|V_{\rm CKM}^{12}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14})\,\epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2}\,|(1+{\rm e}^{\frac{6\pi}{7}\,i})\,\epsilon_d - 2\,{\rm e}^{i\,\alpha}\,\epsilon_u| \\ \frac{1}{2}\,|(1+{\rm e}^{\frac{6\pi}{7}\,i})\,\epsilon_u - 2\,{\rm e}^{i\,\alpha}\,\epsilon_d| & \cos(\frac{\pi}{14})\,\epsilon_u & 1 \end{pmatrix} \tag{3.2}$$

$$|V_{\rm CKM}^{21}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2}\,|(1+{\rm e}^{\frac{6\pi}{7}\,i})\,\epsilon_d - 2\,{\rm e}^{i\,\alpha}\,\epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{\pi}{14})\,\epsilon_d \\ \cos(\frac{\pi}{14})\,\epsilon_u & \frac{1}{2}\,|(1+{\rm e}^{\frac{6\pi}{7}\,i})\,\epsilon_u - 2\,{\rm e}^{i\,\alpha}\,\epsilon_d| & 1 \end{pmatrix} \tag{3.3}$$

$$|V_{\rm CKM}^{22}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2}\,|(1+{\rm e}^{\frac{\pi}{7}\,i})\,\epsilon_d - 2\,{\rm e}^{i\,\alpha}\,\epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7})\,\epsilon_d \\ \frac{1}{2}\,|(1+{\rm e}^{\frac{\pi}{7}\,i})\,\epsilon_u - 2\,{\rm e}^{i\,\alpha}\,\epsilon_d| & \cos(\frac{3\pi}{7})\,\epsilon_u & 1 \end{pmatrix} \tag{3.4}$$

Without loss of generality we have set the representation index j to 1, the group theoretical phase ϕ_u to zero ($m_u = 0$) and the phase ϕ_d to $\frac{2\pi}{14}$ ($m_d = 1$, $n = 14$) for eq. (3.1) and eq. (3.4), while we take it to be $\frac{6\pi}{7}$ ($m_d = 3$, $n = 7$) for eq. (3.2) and eq. (3.3).
Comparing eq. (3.1) to the best fit values of $|V_{ub}|$ and $|V_{td}|$ given in [8] leads to $\epsilon_u \approx 0.0366$ and $\epsilon_d \approx 0.0178$. The phase α is then mainly determined by the values of $|V_{cb}|$ and $|V_{ts}|$. A numerical computation leads to a best fit for $\alpha \approx 4.810$.[3] Furthermore one can calculate $J_{\rm CP}$ in this case:

$$\begin{aligned} J_{\rm CP}^{11} &= \frac{1}{8}\,\sin\left(\frac{\pi}{7}\right)\,\sin\left(\frac{\pi}{14}\right)\,\sin\left(2\,\theta_d\right)\,\sin\left(2\,\theta_u\right)\,\sin\left(\frac{\pi}{14} - \alpha\right) \\ &\approx \frac{1}{2}\,\sin\left(\frac{\pi}{7}\right)\,\sin\left(\frac{\pi}{14}\right)\,\sin\left(\frac{\pi}{14} - \alpha\right)\,\epsilon_u\,\epsilon_d \end{aligned}$$

A similar analysis can be carried out for the three other matrices $V_{\rm CKM}^{12}$, $V_{\rm CKM}^{21}$ and $V_{\rm CKM}^{22}$ with similar results which we have collected in table 1. The value of $J_{\rm CP}$ belonging to $V_{\rm CKM}^{22}$, i.e. $J_{\rm CP}^{22}$, is of the same form as $J_{\rm CP}^{11}$. For $V_{\rm CKM}^{12}$ and $V_{\rm CKM}^{21}$ one finds

$$\begin{aligned} J_{\rm CP}^{12} = J_{\rm CP}^{21} &= -\frac{1}{8}\,\sin\left(\frac{6\,\pi}{7}\right)\,\sin\left(\frac{3\,\pi}{7}\right)\,\sin\left(2\,\theta_d\right)\,\sin\left(2\,\theta_u\right)\,\sin\left(\frac{3\,\pi}{7} - \alpha\right) \\ &\approx -\frac{1}{2}\,\sin\left(\frac{6\,\pi}{7}\right)\,\sin\left(\frac{3\,\pi}{7}\right)\,\sin\left(\frac{3\,\pi}{7} - \alpha\right)\,\epsilon_u\,\epsilon_d \end{aligned}$$

As one can see in table 1, $\epsilon_{u,d}$ have to be larger in case of $V_{\rm CKM}^{22}$, since they are determined by $|V_{cb}|$ and $|V_{ts}|$. In this way the expansion of $\theta_{u,d}$ around $\frac{\pi}{2}$ gets worse and the second order in $\epsilon_{u,d}$ becomes important. This can be seen best in $|V_{us}| \approx 0.2225$ and $|V_{cd}| \approx 0.2225$ which are lowered to 0.2186(5) such that the discrepancy between the experimentally measured

[3] We performed a χ^2 fit of $J_{\rm CP}$ and all elements of $|V_{\rm CKM}|$ excluding the one which is fixed by group theory. Instead of taking the (very small) experimental errors we simply assumed 10% errors for all quantities.

Parameters	V_{CKM}^{11}	V_{CKM}^{12}	V_{CKM}^{21}	V_{CKM}^{22}
ϵ_u	+0.0364	+0.0427	+0.00831	+0.188
ϵ_d	+0.0177	+0.00405	+0.0433	+0.191
α	4.810	2.355	1.764	0.2056

Table 1: Fit results for $\epsilon_{u,d}$ $(\theta_{u,d})$ and the phase α for V_{CKM} with either $|V_{ud}|$, $|V_{us}|$, $|V_{cd}|$ or $|V_{cs}|$ being group theoretically determined.

value and the result of the fit gets larger. However, corrections from higher-dimensional operators and explicit breakings of the residual subgroups can lead to further contributions allowing all data to be fitted successfully.

4. Analysis of the quark sector

In a next step, we construct a viable model at least for the quark sector. The model is viable, if we find a numerical solution which accommodates not only the mixing parameters, but also the quark masses. Due to the strong hierarchy among the quarks this is a non-trivial task, although the number of parameters in the mass matrices M_u and M_d exceeds the number of observables. In the simplest case we assume that all Higgs fields are $\text{SU}(2)_L$ doublets as the Higgs field in the SM.

4.1 D_7 assignments for quarks

Here we present ways to produce the two matrix structures M_4 and M_5 shown in eq. (2.3) and eq. (2.4) with the help of the dihedral group D_7. Choosing D_7 as flavor symmetry leaves us the possibility of either determining $|V_{us}|$ or $|V_{cd}|$ in terms of group theoretical quantities as $\cos(\frac{3\pi}{7})$.

4.1.1 Matrix structure M_4

For M_4, we assign the quarks to

$$Q_1 \sim \underline{\mathbf{1_1}} \; , \; \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \underline{\mathbf{2_1}} \; , \; u_1^c, d_1^c \sim \underline{\mathbf{1_2}} \; , \; u_{2,3}^c, d_{2,3}^c \sim \underline{\mathbf{1_1}} \tag{4.1}$$

under D_7. We assume that the theory contains Higgs doublet fields transforming as $\underline{\mathbf{1_1}}$ and $\underline{\mathbf{2_1}}$, which we call H_s and $H_{1,2}$. As the relation between the mixing parameters of V_{CKM} and the group theoretical indices only arises, if the flavor symmetry D_7 is broken down to a subgroup $Z_2 =< \text{B}\,\text{A}^{m_u} >$ by fields which couple to up quarks, while it is broken down to $Z_2 =< \text{B}\,\text{A}^{m_d} >$ with $m_d \neq m_u$ by fields coupling to down quarks, we need an extra symmetry to perform this separation. In the SM this can be achieved by a $Z_2^{(aux)}$ symmetry:

$$d_i^c \;\rightarrow\; -d_i^c \quad \text{and} \quad H_s^d \;\rightarrow\; -H_s^d \; , \; H_i^d \;\rightarrow\; -H_i^d \tag{4.2}$$

while all other fields Q_i, u_i^c, H_s^u and $H_{1,2}^u$ are invariant under $Z_2^{(aux)}$. In principle also a Higgs field transforming as $\underline{\mathbf{1_2}}$ under D_7 could couple directly to the quarks. However, if this field acquires a non-vanishing VEV, its VEV breaks the residual Z_2 generated by $<\mathrm{B}\,\mathrm{A}^m>$. The matrices are of the form:

$$M_u = \begin{pmatrix} 0 & y_1^u \langle H_s^u \rangle^\star & y_2^u \langle H_s^u \rangle^\star \\ y_3^u \langle H_1^u \rangle^\star & y_4^u \langle H_1^u \rangle^\star & y_5^u \langle H_1^u \rangle^\star \\ -y_3^u \langle H_2^u \rangle^\star & y_4^u \langle H_2^u \rangle^\star & y_5^u \langle H_2^u \rangle^\star \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} 0 & y_1^d \langle H_s^d \rangle & y_2^d \langle H_s^d \rangle \\ y_3^d \langle H_2^d \rangle & y_4^d \langle H_2^d \rangle & y_5^d \langle H_2^d \rangle \\ -y_3^d \langle H_1^d \rangle & y_4^d \langle H_1^d \rangle & y_5^d \langle H_1^d \rangle \end{pmatrix}$$

where $y_i^{u,d}$ denote Yukawa couplings. The VEV structure is taken to be:

$$\langle H_s^{d,u} \rangle > 0\ , \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d\ , \quad \langle H_1^u \rangle = v_u\, \mathrm{e}^{-\frac{3\,\pi\,i}{7}} \quad \text{and} \quad \langle H_2^u \rangle = v_u\, \mathrm{e}^{\frac{3\,\pi\,i}{7}}$$

with $v_d > 0$ and $v_u > 0$. The VEVs are required to be real apart from the phase $\pm\frac{3\pi}{7}$ which is necessary for the correct breaking to the desired subgroup of D_7. The parameters $A, B, \ldots$ shown in eq. (2.3) and eq. (2.4) can be written in terms of Yukawa couplings and VEVs:

$$\begin{aligned} &A_u = y_1^u \langle H_s^u \rangle\ , \quad B_u = y_2^u \langle H_s^u \rangle\ , \quad C_u = y_3^u v_u \mathrm{e}^{-\frac{3\,\pi\,i}{7}}, \quad D_u = y_4^u v_u \mathrm{e}^{-\frac{3\,\pi\,i}{7}}, \quad E_u = y_5^u v_u \mathrm{e}^{-\frac{3\,\pi\,i}{7}}, \\ &A_d = y_1^d \langle H_s^d \rangle, \quad B_d = y_2^d \langle H_s^d \rangle, \quad C_d = y_3^d v_d, \quad D_d = y_4^d v_d, \quad E_d = y_5^d v_d \end{aligned}$$

together with $\phi_u = \frac{6\pi}{7}$ $(m_u = 3)$, $\phi_d = 0$ $(m_d = 0)$ and $\mathrm{j} = 1$. The preserved Z_2 subgroups are generated by $\mathrm{B}\,\mathrm{A}^3$ and B. As we have not fixed the ordering of the mass eigenvalues, the question which of the elements of V_{CKM} is determined by group theoretical quantities to be $\cos(\frac{3\pi}{7})$ cannot be answered at this point.

4.1.2 Matrix structure M_5

For the case of M_5 we can assign the quarks to:

$$Q_1, u_1^c, d_1^c \sim \underline{\mathbf{1}}_{\mathbf{1}}\ , \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}, \begin{pmatrix} u_2^c \\ u_3^c \end{pmatrix}, \begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \tag{4.3}$$

under D_7. We then need five Higgs fields for each sector, i.e. for the up and the down quarks. These transform as

$$\begin{aligned} &H_s^u \sim (\underline{\mathbf{1}}_{\mathbf{1}}, +1)\ , \quad &\begin{pmatrix} H_1^u \\ H_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, +1)\ , \quad &\begin{pmatrix} h_1^u \\ h_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{2}}, +1) \\ &H_s^d \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -1)\ , \quad &\begin{pmatrix} H_1^d \\ H_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, -1)\ , \quad &\begin{pmatrix} h_1^d \\ h_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{2}}, -1) \end{aligned}$$

where we again assumed the existence of an extra $Z_2^{(aux)}$ symmetry. The mass matrices are in terms of Yukawa couplings and VEVs:

$$M_u = \begin{pmatrix} y_1^u \langle H_s^u \rangle^\star & y_2^u \langle H_1^u \rangle^\star & y_2^u \langle H_2^u \rangle^\star \\ y_3^u \langle H_1^u \rangle^\star & y_5^u \langle h_1^u \rangle^\star & y_4^u \langle H_s^u \rangle^\star \\ y_3^u \langle H_2^u \rangle^\star & y_4^u \langle H_s^u \rangle^\star & y_5^u \langle h_2^u \rangle^\star \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_1^d \langle H_s^d \rangle & y_2^d \langle H_2^d \rangle & y_2^d \langle H_1^d \rangle \\ y_3^d \langle H_2^d \rangle & y_5^d \langle h_2^d \rangle & y_4^d \langle H_s^d \rangle \\ y_3^d \langle H_1^d \rangle & y_4^d \langle H_s^d \rangle & y_5^d \langle h_1^d \rangle \end{pmatrix}$$

The VEV structure is assumed to be:

$$\begin{aligned} &\langle H_s^{d,u} \rangle > 0\ , \quad &\langle H_1^d \rangle = \langle H_2^d \rangle = v_d\ , \quad &\langle h_1^d \rangle = \langle h_2^d \rangle = w_d\ , \\ &\langle H_1^u \rangle = v_u\, \mathrm{e}^{-\frac{3\,\pi\,i}{7}}\ , \quad &\langle H_2^u \rangle = v_u\, \mathrm{e}^{\frac{3\,\pi\,i}{7}}\ , \quad &\langle h_1^u \rangle = w_u\, \mathrm{e}^{-\frac{6\,\pi\,i}{7}} \quad \text{and} \quad \langle h_2^u \rangle = w_u\, \mathrm{e}^{\frac{6\,\pi\,i}{7}} \end{aligned}$$

with $v_{d,u} > 0$ and $w_{d,u} > 0$. As above we only consider real values for the VEVs apart from the phases which are required in order to break down to a certain subgroup of D_7. Compared to the form of M_5 (see eq. (2.3) and eq. (2.4)) we see that the parameters $A, B, \ldots$ are given by:

$$\begin{array}{lllll} A_u = y_1^u \langle H_s^u \rangle, & B_u = y_3^u v_u \mathrm{e}^{-\frac{3\pi i}{7}}, & C_u = y_2^u v_u \mathrm{e}^{-\frac{3\pi i}{7}}, & D_u = y_5^u w_u \mathrm{e}^{-\frac{6\pi i}{7}}, & E_u = y_4^u \langle H_s^u \rangle, \\ A_d = y_1^d \langle H_s^d \rangle, & B_d = y_3^d v_d, & C_d = y_2^d v_d, & D_d = y_5^d w_d, & E_d = y_4^d \langle H_s^d \rangle \end{array}$$

together with $\phi_u = \frac{6\pi}{7}$ ($m_u = 3$), $\phi_d = 0$ ($m_d = 0$) and j = k = 1. Therefore the preserved subgroups are again $Z_2 =< \mathrm{B\,A}^3 >$ and $Z_2 =< \mathrm{B} >$.
Note that the shown assignments are not unique, since it is also possible to use another two-dimensional representation instead of $\underline{\mathbf{2}}_{\mathbf{1}}$ for the fermions. Obviously, then also the transformation properties of the Higgs fields have to be changed accordingly. From the viewpoint of unification the second assignment in which the left-handed as well as the left-handed conjugate fields transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ is more desirable. However in this case we need at least five Higgs fields in each sector transforming as $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{2}}_{\mathbf{i}}$, $\underline{\mathbf{2}}_{\mathbf{j}}$ with i $\neq$ j in order to arrive at the matrix structure M_5. Since we want to show the minimal model, we constrain ourselves to the case of M_4 in the following numerical study and the analysis of the corresponding Higgs potential. We only give a numerical solution for the second matrix structure M_5.

4.2 Numerical analysis of quark masses and mixing angles

4.2.1 Matrix structure M_4

For our numerical results we take all VEVs to have the same absolute value of 61.5 GeV which equals the electroweak scale 174 GeV divided by $\sqrt{8}$, because our complete model includes eight Higgs fields.[4] The Yukawa couplings are taken to be

$$\begin{array}{lll} y_1^u{=}1.07967 \cdot \mathrm{e}^{i(-2.17704)}, & y_2^u{=}2.55955 \cdot \mathrm{e}^{i(1.41609)}, & y_3^u{=}1.9546 \cdot 10^{-5} \cdot \mathrm{e}^{i(2.43366)}, \\ y_4^u{=}3.89557 \cdot 10^{-2} \cdot \mathrm{e}^{i(-2.28452)}, & y_5^u{=}7.47229 \cdot 10^{-2} \cdot \mathrm{e}^{i(1.2469)}, & \\ y_1^d{=}2.52251 \cdot 10^{-2} \cdot \mathrm{e}^{i(3.00267)}, & y_2^d{=}3.92611 \cdot 10^{-2} \cdot \mathrm{e}^{i(-2.29202)}, & y_3^d{=}6.20874 \cdot 10^{-4} \cdot \mathrm{e}^{i(-0.54014)}, \\ y_4^d{=}8.95471 \cdot 10^{-5} \cdot \mathrm{e}^{i(-2.13972)}, & y_5^d{=}1.04917 \cdot 10^{-4} \cdot \mathrm{e}^{i(-1.59912)} & \end{array}$$

All quark masses are fitted to the central values at M_Z found in [10]. For V_{CKM}, we find:

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.97492 & 0.2225 & 3.95 \times 10^{-3} \\ 0.2224 & 0.97404 & 42.23 \times 10^{-3} \\ 8.11 \times 10^{-3} & 41.64 \times 10^{-3} & 0.9991 \end{pmatrix}$$

and $J_{\mathrm{CP}} = 3.09 \times 10^{-5}$. All these values are within a 10 % error range [8] with $|V_{us}|$ fixed to be $\cos(\frac{3\pi}{7}) = 0.2225$. Due to the ordering of the eigenvalues the mass of the strange

[4] The additional two Higgs fields which do not couple to the fermions directly, are necessary in order to break accidental symmetries present in the Higgs potential which we discuss in section 5. The equality of the VEVs is motivated by our numerical study of the Higgs potential.

as well as the one of the up quark is determined by $\sqrt{2}\,|C_d|$ and $\sqrt{2}\,|C_u|$, respectively. They therefore correspond to the eigenvalue $(c-d)$ in the language of section 2. The Yukawa couplings lie in the range $10^{-5}\ldots 1$ due to the strong hierarchy of the quark masses. However this can be explained by the Froggatt-Nielsen (FN) mechanism [11]. For example, assuming the FN field ϑ with $q_{\mathrm{FN}}(\vartheta) = -1$ and taking $q_{\mathrm{FN}}(Q_1) = +1$, $q_{\mathrm{FN}}(Q_{2,3}) = +2$, $q_{\mathrm{FN}}(d^c_{1,2,3}) = 0$, $q_{\mathrm{FN}}(u^c_1) = +1$ and $q_{\mathrm{FN}}(u^c_{2,3}) = -1$ under $\mathrm{U}(1)_{\mathrm{FN}}$ allows all Yukawa couplings to be of natural order.

4.2.2 Matrix structure M_5

For the second matrix structure M_5, we also performed a numerical study with the mass matrix structure given above and found the following possible values for the parameters $A_{u,d}, B_{u,d}, \ldots$:

$$
\begin{aligned}
A_u &= 40.40221\cdot \mathrm{e}^{i\,(0.185452)}\,, & B_u &= 0.238084\cdot \mathrm{e}^{i\,(-2.99845)}\,, & C_u &= 117.4875\cdot \mathrm{e}^{i\,(-0.234118)}\,,\\
D_u &= 0.420584\cdot \mathrm{e}^{i\,(-3.13931)}\,, & E_u &= 0.984542\cdot \mathrm{e}^{i\,(-0.849532)}\,, & &\\
A_d &= 2.233447\cdot \mathrm{e}^{i\,(-1.91017)}\,, & B_d &= 0.051223\cdot \mathrm{e}^{i\,(-3.05165)}\,, & C_d &= 1.271448\cdot \mathrm{e}^{i\,(-0.751605)}\,,\\
D_d &= 0.058343\cdot \mathrm{e}^{i\,(-2.41411)}\,, & E_d &= 0.056221\cdot \mathrm{e}^{i\,(-2.37708)}\,. & &
\end{aligned}
$$

All values are given in GeV. The phases $\phi_{u,d}$ can be chosen to be $\phi_u = \frac{6\pi}{7}$ and $\phi_d = 0$. Again, the quark masses match the central values given in [10], while the absolute values of V_{CKM} are:

$$
|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.97489 & 0.2226 & 3.95\times 10^{-3} \\ 0.2225 & 0.97401 & 42.23\times 10^{-3} \\ 8.11\times 10^{-3} & 41.64\times 10^{-3} & 0.9991 \end{pmatrix}
$$

together with $J_{\mathrm{CP}} = 3.09\times 10^{-5}$. They agree quite well with the experimental results. Note here that this time not $|V_{us}|$, but now $|V_{cd}|$ is given in terms of the group theoretical indices, i.e. $|V_{cd}| = \cos(\frac{3\pi}{7}) = 0.2225$. This is due to the fact that the eigenvalue $(c-d)$ introduced in section 2 is given by m_c in the up quark and by m_d in the down quark sector. These masses can be expressed in a simple way in terms of the parameters $D_{u,d}$ and $E_{u,d}$, namely $m_c = |D_u - E_u\,\mathrm{e}^{-i\,\phi_u\,\mathrm{k}}|$ and $m_d = |D_d - E_d\,\mathrm{e}^{i\,\phi_d\,\mathrm{k}}|$ with $\phi_u = \frac{6\pi}{7}$, $\phi_d = 0$ and $\mathrm{k} = 1$. Also here the hierarchy among the parameters $A_{u,d}, B_{u,d}, \ldots$ may not be explained by the flavor symmetry $D_7 \times {Z_2}^{(aux)}$ alone. However, we can again assume the existence of an additional $\mathrm{U}(1)_{\mathrm{FN}}$ symmetry.

5. Higgs sector

In this section, the Higgs sector belonging to the first numerical example given in section 4.1.1 is discussed. As already mentioned above, we concentrate on a multi-Higgs doublet potential. We are aware of the fact that such multi-Higgs doublet models usually suffer from the problem that large FCNCs are induced by the additional Higgs fields. However, as a proof of principle that we can produce our required VEV configuration the

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consideration of such a setup seems to be reasonable. The minimal number of fields needed in order to produce the fermion mass matrices is 2×3, H^d_s, $H^d_{1,2}$ and H^u_s, $H^u_{1,2}$.
We first construct the three Higgs doublet potential with Higgs fields $H_s \sim \underline{\mathbf{1}}_{\mathbf{1}}$ and $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$.
The potential has the form:[5]

$$
\begin{aligned}
V_3(H_s, H_i) = & -\mu_s^2 \, H_s^\dagger \, H_s - \mu_D^2 \sum_{i=1}^{2} H_i^\dagger \, H_i + \lambda_s \left(H_s^\dagger \, H_s \right)^2 + \lambda_1 \left(\sum_{i=1}^{2} H_i^\dagger \, H_i \right)^2 \\
& + \lambda_2 \left(H_1^\dagger \, H_1 - H_2^\dagger \, H_2 \right)^2 + \lambda_3 \, |H_1^\dagger \, H_2|^2 \\
& + \sigma_1 \left(H_s^\dagger H_s \right) \left(\sum_{i=1}^{2} H_i^\dagger H_i \right) + \left\{ \sigma_2 \left(H_s^\dagger H_1 \right) \left(H_s^\dagger H_2 \right) + \text{h.c.} \right\} + \sigma_3 \sum_{i=1}^{2} |H_s^\dagger H_i|^2
\end{aligned}
\tag{5.1}
$$

As already shown in [12] and also mentioned in [13], this potential has an additional U(1) symmetry, i.e. there exists a further U(1) in the potential apart from the $\mathrm{U}(1)_Y$. This further symmetry is necessarily broken by our desired VEV structure such that a massless Goldstone boson appears in the Higgs spectrum which is not eaten by a gauge boson. This problem cannot be solved by taking into account the whole potential for all six Higgs fields. Therefore we have to enlarge the Higgs sector by further fields in order to create new D_7-invariant couplings which break this accidental symmetry explicitly. We find that this can be done in the simplest way by adding two Higgs fields transforming as $\underline{\mathbf{2}}_{\mathbf{2}}$ under D_7. Due to their transformation properties they do not directly couple to the fermions (see section 4.1.1). The complete model then contains eight Higgs doublet fields

$$
\begin{aligned}
& H^u_s \sim (\underline{\mathbf{1}}_{\mathbf{1}}, +1) \, , && \begin{pmatrix} H^u_1 \\ H^u_2 \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, +1) \, , \\
& H^d_s \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -1) \, , && \begin{pmatrix} H^d_1 \\ H^d_2 \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, -1) \quad \text{and} \quad \begin{pmatrix} \chi^d_1 \\ \chi^d_2 \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{2}}, -1)
\end{aligned}
\tag{5.2}
$$

under $D_7 \times Z_2^{(aux)}$. The potential consists of three parts:

$$
V = V_u + V_d + V_{\text{mixed}} \tag{5.3}
$$

where V_u denotes the part of the potential which only contains Higgs fields coupling to the up quarks, V_d contains the five Higgs fields which have a non-vanishing $Z_2^{(aux)}$ charge, while V_{mixed} consists of all other terms. The explicit form of the potential is given in appendix C. The VEV structure of the fields $H^{d,u}_s$ and $H^{d,u}_{1,2}$ is determined by our desire to break down to two distinct Z_2 subgroups in the up and the down quark sector (see section 4.1.1):

$$
\langle H^{d,u}_s \rangle > 0 \, , \;\; \langle H^d_1 \rangle = \langle H^d_2 \rangle = v_d \, , \;\; \langle H^u_1 \rangle = v_u \, \mathrm{e}^{-\frac{3\,\pi\,i}{7}} \quad \text{and} \quad \langle H^u_2 \rangle = v_u \, \mathrm{e}^{\frac{3\,\pi\,i}{7}}
$$

[5]Note that σ_2 is complex, but it can be made real by appropriate redefinition of the field H_s, for example.

with $v_d > 0$ and $v_u > 0$. In contrast to this, the VEV structure of the fields $\chi^d_{1,2}$ is not fixed in this way. However, in order to preserve the Z_2 subgroup generated by B not only through the VEVs of the fields H^d_s and $H^d_{1,2}$, but also by the VEVs of the fields $\chi^d_{1,2}$, $\langle \chi^d_1 \rangle = \langle \chi^d_2 \rangle > 0$ will be assumed (see section 2).
We proceed in the following way in order to find a minimum of this potential which allows for our choice of VEVs: first we treat V_u and V_d separately to find a viable solution for these two parts of the potential. Thereby, we can allow all parameters in the potential V_d to be real, as the VEVs of the corresponding Higgs fields are also real. Since V_u suffers from the above mentioned accidental U(1) symmetry, we find a fourth massless particle in the Higgs mass spectrum. In a second step we add as many terms as necessary from V_{mixed} to get a minimum of the whole potential V which does not have more than the usual three Goldstone bosons. It turns out that it is sufficient to take into account three terms in addition to V_u and V_d to get a viable solution. The terms are of the form:

$$\kappa_2 \left(H^{u\dagger}_s H^d_s\right)^2 + \kappa_5 \left(\sum_{i=1}^{2} H^{u\dagger}_i H^d_i\right)^2 + \kappa_{19} \left(H^{u\dagger}_s H^d_s\right) \left(\sum_{i=1}^{2} H^{u\dagger}_i H^d_i\right) + \text{h.c.} \subset V_{\text{mixed}}$$

All VEVs are taken to have the same absolute value, since this considerably simplifies the search for a numerical solution, as a fine-tuning of the parameters in the Higgs potential is avoided. We find that the resulting Higgs masses are usually in between 50 and 500 GeV. These values are either not favored by the constraints coming from FCNCs or already excluded by direct searches. There are two reasons for the too low Higgs masses: on the one hand V_u contains an accidental symmetry and on the other hand all mass parameters of the potential are chosen to be of natural order, i.e. to be around the electroweak scale. Additionally, all quartic couplings of the potential must be perturbative. However, as already mentioned above, this model is not intended to be fully realistic. Adding D_7 breaking soft masses to the potential might allow to push the masses of the additional Higgs particles above 10 TeV.
The rest of the discussion of the potential is relegated to appendix C where we present a numerical solution for the parameters of the Higgs potential and the resulting Higgs masses.

6. Ways to generate θ_C only

In the preceding sections we confined ourselves to cases in which all mixing angles can be reproduced at tree level. Therefore we only discussed the matrix structures M_4 and M_5 of eq. (2.3) and eq. (2.4). However, θ^q_{13} and θ^q_{23} are roughly an order of magnitude smaller than the Cabibbo angle $\theta_C \equiv \theta^q_{12}$ which gives reason for also considering matrix structures which lead to only $\theta_C \neq 0$ at LO. For this a block matrix structure (with correlated elements), which we introduced in eq. (2.2), is suitable. Such a structure can be achieved in at least two different ways: *a.*) we can simply omit some of the Higgs fields which are in principle allowed a VEV in order to arrive at the zero elements of the mass matrix; *b.*) we can demand that the preserved subgroup is not just a Z_2 symmetry, but a dihedral group D_q with $q > 1$. For case *a.*) the simplest example is probably the one in which we take the

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same field assignments as in the case of the matrix structure M_5 (see eq. (4.3)), but we omit the Higgs fields $H^{u,d}_{1,2}$ transforming as $\underline{\mathbf{2}}_{\mathbf{1}}$. The second case *b.*) cannot be maintained with the flavor group D_7 which we used throughout this work, since it only contains Z_q groups as subgroups, but no dihedral ones D_q, $q > 1$. Therefore we have to consider the group D_{14} instead. One possibility is to break D_{14} down to its subgroup $D_2 =< \mathrm{A}^7, \mathrm{B}\,\mathrm{A}^m >$ ($m = 0, 1, \ldots, 6$) in order to reproduce a matrix of block structure. We assign the quarks to

$$Q_1, u^c_1, d^c_1 \sim \underline{\mathbf{1}}_{\mathbf{1}} \;\; , \;\; \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}, \begin{pmatrix} u^c_2 \\ u^c_3 \end{pmatrix}, \begin{pmatrix} d^c_2 \\ d^c_3 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

under D_{14}. According to the Kronecker products

$$\underline{\mathbf{1}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}} = \underline{\mathbf{2}}_{\mathbf{1}} \quad \text{and} \quad \underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}} = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{2}}$$

the Higgs fields which can in principle couple to form D_{14}-invariants have to transform as $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$, $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$. However, $\underline{\mathbf{1}}_{\mathbf{2}}$ is not allowed a VEV and the representation index j of $\underline{\mathbf{2}}_{\mathbf{j}}$ has to be even for preserving a D_2 subgroup. Therefore we take

$$H^u_s \sim \underline{\mathbf{1}}_{\mathbf{1}} \; , \; \begin{pmatrix} H^u_1 \\ H^u_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \; , \; H^d_s \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad \text{and} \quad \begin{pmatrix} H^d_1 \\ H^d_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}$$

(with implicit $Z_2^{(aux)}$ assignment as above) and arrive at matrices which are exactly of the same form as in case *a.*), if we assume the VEVs to be

$$\langle H^{u,d}_s \rangle > 0 \; , \;\; \langle H^u_1 \rangle = w_u \, \mathrm{e}^{-\frac{6\,\pi\,i}{7}} \;\; , \;\; \langle H^u_2 \rangle = w_u \, \mathrm{e}^{\frac{6\,\pi\,i}{7}} \;\; , \;\; \langle H^d_1 \rangle = \langle H^d_2 \rangle = w_d$$

The subgroups D_2 which are preserved by the VEVs are then of the announced form with $m_u = 6$ for the up quarks and $m_d = 0$ for down quarks.

7. Numerical analysis of V_{MNS}

A similar analysis as done in the case of V_{CKM} can also be carried out for the lepton mixing matrix V_{MNS}. We assume that the neutrinos are Dirac particles as all the other fermions and that they have the same ordering as the other fermions, i.e. the neutrino mass spectrum is normally ordered. This allows us to use the matrix structures found in appendix A also for V_{MNS}. Since the entries of V_{MNS} are not strongly restricted by experiments [14] (at $3\,\sigma$):

$$\left| V^{(\mathrm{range})}_{\mathrm{MNS}} \right| = \begin{pmatrix} 0.79 - 0.88 & 0.47 - 0.61 & < 0.20 \\ 0.19 - 0.52 & 0.42 - 0.73 & 0.58 - 0.82 \\ 0.20 - 0.53 & 0.44 - 0.74 & 0.56 - 0.81 \end{pmatrix} \tag{7.1}$$

there are several more possibilities to accommodate the various matrix elements regarding the choice of the group index n, and the values m_l, m_ν and j. However, as we intend to build a model which includes quarks as well as leptons, we stick to the selected values of n, $n = 7$, $n = 14$, which fit the CKM matrix elements of the $1-2$ sub-block best for small n. We check element by element of V_{MNS} whether we can put it into the form $|\cos(\frac{l\pi}{7})|$ where $l =$

Element (ij)	Possible cosines
(21)	$\cos(\frac{3\pi}{7})$ (≈ 0.2225), $\cos(\frac{5\pi}{14})$ (≈ 0.4339)
(22)	$\cos(\frac{5\pi}{14})$ (≈ 0.4339), $\cos(\frac{2\pi}{7})$ (≈ 0.6235)
(23)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235), $\cos(\frac{3\pi}{14})$ (≈ 0.7818)
(31)	$\cos(\frac{3\pi}{7})$ (≈ 0.2225), $\cos(\frac{5\pi}{14})$ (≈ 0.4339)
(32)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235)
(33)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235), $\cos(\frac{3\pi}{14})$ (≈ 0.7818)

Table 2: Possibilities for the group theoretically determined element in $V_{\rm MNS}$.

$0, 1, 2, \ldots, 6$ or $|\cos(\frac{l\pi}{14})|$ with $l = 0, 1, 2, \ldots, 13$. According to eq. (7.1) all elements of the second and third row can be approximated by a cosine of such a form. We take into account all possibilities shown in table 2 and perform a numerical fit of the mixing angles θ_{12}, θ_{13} and θ_{23}. In the fit procedure we compute the sines of the three mixing angles and compare these to the best fit values, which are $\sin^2(\theta_{23}^{bf}) = 0.5$, $\sin^2(\theta_{12}^{bf}) = 0.3$ and $\sin^2(\theta_{13}^{bf}) = 0$ [15].[6] Again, we replace the experimentally allowed $2\,\sigma$ or $3\,\sigma$ ranges by 10% ranges (around the best fit value). For $\sin^2(\theta_{13})$ we consider two possible upper bounds: $\sin^2(\theta_{13}) \leq 0.025$ which corresponds to the $2\,\sigma$ bound [15] and a much more loose bound $\sin^2(\theta_{13}) \leq 0.1$ being even larger than the $4\,\sigma$ bound [15]. This is done, since the numerical study showed that loosening the bound on $\sin^2(\theta_{13})$ leads to several more solutions. Our results for $\sin^2(\theta_{13}) \leq 0.1$ are summarized in table 3 where we display the numerical values for θ_l, θ_ν and $\alpha = \beta_l - \beta_\nu$ together with the resulting mixing angles and the (Dirac) CP phase δ.
One can observe the following: There are some cosines listed in table 2 for which no fit with $\chi^2 < 1$ has been found. In all these cases the value of the fixed $V_{\rm MNS}$ element lies almost outside the ranges shown in eq. (7.1), e.g. for the (23) element the possible cosines are $\cos(\frac{2\pi}{7}) \approx 0.6235$ and $\cos(\frac{3\pi}{14}) \approx 0.7818$ with the first being quite close to the lower bound (0.58) and the second one close to the upper one (0.82) of the allowed range. More precisely, the form of $|V_{\rm mix}^{23,33}|$ reveals that at least in these cases it is hardly possible to reconcile the two experimental constraints $\tan(\theta_{23})$ being close to 1 and $\sin(\theta_{13})$ being small. Furthermore, one observes that in all cases the CP phase δ is trivial, i.e. 0 or π with a numerical precision of $\mathcal{O}(10^{-6})$. Therefore $J_{\rm CP}$ always vanishes. In order to understand this result, we have a look at the formulae given for $V_{\rm mix}^{21}$, $V_{\rm mix}^{22}$, $V_{\rm mix}^{31}$ and $V_{\rm mix}^{32}$ in appendix A. As a common feature the (13) element of the mixing matrix is given by

$$\frac{1}{2}\,[-(1 + {\rm e}^{-i\,(\phi_l - \phi_\nu)\,{\rm j}})\,\sin(\theta_l)\,\cos(\theta_\nu) + 2\,{\rm e}^{i\,\alpha}\,\cos(\theta_l)\,\sin(\theta_\nu)] \tag{7.2}$$

In all cases, θ_l and θ_ν are predominantly determined by one element of the first row and the third column of $V_{\rm MNS}$, respectively. Then α can be used in order to minimize the absolute

[6] Note that these best fit values are not presented in the same global analysis as the above mentioned allowed $3\,\sigma$ ranges for the elements of $V_{\rm MNS}$. Nevertheless the deviations are very small such that we do not consider this to lead to a major difference in our numerical analysis.

Element	Cosine	θ_l	θ_ν	α	$\sin^2(\theta_{12})$	$\sin^2(\theta_{23})$	$\sin^2(\theta_{13})$	δ
(21)	$\cos(\frac{3\pi}{7})$	0.9790	0.7881	4.937	0.2957	0.5085	7.037×10^{-2}	$\sim \pi$
	$\cos(\frac{5\pi}{14})$	1.1829	0.6725	5.161	0.3001	0.4999	6.173×10^{-3}	~ 0
(22)	$\cos(\frac{5\pi}{14})$	–	–	–	–	–	–	–
	$\cos(\frac{2\pi}{7})$	0.7728	0.4486	5.386	0.2999	0.4996	6.668×10^{-3}	$\sim \pi$
(23)	$\cos(\frac{2\pi}{7})$	–	–	–	–	–	–	–
	$\cos(\frac{3\pi}{14})$	–	–	–	–	–	–	–
(31)	$\cos(\frac{3\pi}{7})$	0.9790	0.7881	4.937	0.2957	0.4915	7.037×10^{-2}	~ 0
	$\cos(\frac{5\pi}{14})$	1.1829	0.6725	5.161	0.3001	0.5001	6.173×10^{-3}	$\sim \pi$
(32)	$\cos(\frac{2\pi}{7})$	0.7728	0.4486	5.386	0.2999	0.5004	6.668×10^{-3}	~ 0
(33)	$\cos(\frac{2\pi}{7})$	–	–	–	–	–	–	–
	$\cos(\frac{3\pi}{14})$	–	–	–	–	–	–	–

Table 3: Numerical results for $V_{\rm MNS}$ in case of $\sin^2(\theta_{13}) \leq 0.1$ and 10% errors for the other two sine squares. δ is given with a precision of $\mathcal{O}(10^{-6})$.

value of the (13) element of $V_{\rm MNS}$. A minimization with respect to α shows

$$\alpha = -(\phi_l - \phi_\nu)\,\frac{\rm j}{2} + \pi\, y = -\frac{\pi}{n}\,(m_l - m_\nu)\,{\rm j} + \pi\, y \quad \text{with} \quad y \in \mathbb{Z}_0 \tag{7.3}$$

The minimum value for $|\sin(\theta_{13})|$ is then $|\cos((\phi_l - \phi_\nu)\,\frac{\rm j}{2})\,\sin(\theta_l)\,\cos(\theta_\nu) + (-1)^{y+1}\cos(\theta_l)\,\sin(\theta_\nu)|$. However, in all cases the expression is only minimized for $y = 0, 2, \ldots$, as the involved sines and cosines are all positive. As $J_{\rm CP}$ is proportional to $\sin((\phi_l - \phi_\nu)\,\frac{\rm j}{2} + \alpha)$, it is zero for the calculated value of α. Therefore δ must be either 0 or π. Additionally, we found an explanation for the values of α shown in table 3 given in terms of the group theoretical quantities, i.e. $2\pi - \frac{3\pi}{7} \approx 4.937$, $2\pi - \frac{5\pi}{14} \approx 5.161$ and $2\pi - \frac{2\pi}{7} \approx 5.386$. As a last observation we report that there exist similarities among the different cases, e.g. fixing the (21) element to be $\cos(\frac{3\pi}{7})$ is similar to fixing the (31) element to the same value. The cases coincide concerning the fit values of θ_l, θ_ν and α and the resulting mixing angles $\sin^2(\theta_{12})$ and $\sin^2(\theta_{13})$ (up to $\mathcal{O}(10^{-6})$), whereas $\sin^2(\theta_{23})$ and δ are shifted. This can be understood, since the mixing matrices are related through the interchange of the second and third row.

Using the 2σ bound $\sin^2(\theta_{13}) \leq 0.025$ no solution with $\chi^2 < 1$ is found in the cases in which the (21) or the (31) element is fixed to the value $\cos(\frac{3\pi}{7})$, since the values for $\sin^2(\theta_{13})$ shown in table 3 are quite large. For the other configurations we again find viable fits in which the values θ_l, θ_ν and α are very similar to the ones given in table 3.

Apart from studying how well one can accommodate the experimentally allowed ranges, it is also interesting to see whether one can reproduce some special mixing pattern in the lepton sector. In the following we discuss the TBM scenario which has initially been discussed in [16], since all elements of the lepton mixing matrix can be written in terms of

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Element (ij)	Possible cosines
(11)	$\cos(\frac{3\pi}{14})$ (≈ 0.7818)
(12)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235)
(21)	$\cos(\frac{5\pi}{14})$ (≈ 0.4339)
(22)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235)
(31)	$\cos(\frac{5\pi}{14})$ (≈ 0.4339)
(32)	$\cos(\frac{2\pi}{7})$ (≈ 0.6235)

Table 4: Possibilities for the group theoretically determined element in V_{MNS}, if TBM is assumed to be the best fit.

fractions of square roots $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{6}}$:

$$V_{\text{MNS}}^{\text{TBM}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{7.4}$$

corresponding to sines of the mixing angles:

$$\sin^2(\theta_{23}^{\text{TBM}}) = \frac{1}{2}\ , \quad \sin^2(\theta_{12}^{\text{TBM}}) = \frac{1}{3} \quad \text{and} \quad \sin^2(\theta_{13}^{\text{TBM}}) = 0\ .$$

The uncertainty in the mixing matrix elements is taken to be 10 %, i.e. the fixed element given by cosine $|\cos(\frac{l\pi}{7})|$ for $l = 0, 1, 2, \ldots, 6$ or $|\cos(\frac{l\pi}{14})|$ with $l = 0, 1, 2, \ldots, 13$ should lie in one of the ranges:

$$V_{\text{MNS}}^{\text{TBM (range)}} = \begin{pmatrix} 0.73 - 0.90 & 0.52 - 0.64 & < 0.20 \\ 0.37 - 0.45 & 0.52 - 0.64 & 0.64 - 0.78 \\ 0.37 - 0.45 & 0.52 - 0.64 & 0.64 - 0.78 \end{pmatrix} \tag{7.5}$$

The bound on the (13) element is taken to be the same as in eq. (7.1). As shown in table 4, the elements (11) and (12) can now be described by a cosine of the announced form, while we find less possibilities for the other elements compared to the case of the experimentally allowed range, see table 2. The numerical analysis is analogous to the one above. The results are very similar apart from the case in which the (11) element is determined by group theory. Therefore, we focus on the discussion of this case. First of all, we find that θ_l can take values in a certain range instead of being fixed to a single value. All of them lead to the same mixing angles. The same is true for α which varies between 0 and 2π. This is related to the fact that we do not fit the CP phase δ (or equivalently the Jarlskog invariant J_{CP}). As a result J_{CP} can take any value in the range $(-5.776 \ldots 5.776) \times 10^{-2}$. We observe that θ_ν is fixed by the fit of $\sin^2(\theta_{12})$ and $\sin^2(\theta_{13})$. Fitting them at the same time leads, unfortunately, to a too large value for $\sin^2(\theta_{13})$ (see table 5). The allowed range for θ_l can then be found analytically under the assumption that $\sin^2(\theta_{23}) = \frac{1}{2}$, since in this case the (23) and (33) element of V_{MNS} have to be equal. Equating the expressions

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Element	Cosine	θ_l	θ_ν	α	$\sin^2(\theta_{12})$	$\sin^2(\theta_{23})$	$\sin^2(\theta_{13})$	δ
(11)	$\cos(\frac{3\pi}{14})$	$0.4396-1.131$	1.139	$\in[0,2\pi)$	0.3441	0.5000	6.808×10^{-2}	$\in[\sim0,\sim2\pi)$
(12)	$\cos(\frac{2\pi}{7})$	–	–	–	–	–	–	–
(21)	$\cos(\frac{5\pi}{14})$	1.132	0.6697	5.161	0.3334	0.5000	1.968×10^{-3}	~0
(22)	$\cos(\frac{2\pi}{7})$	0.8235	0.4557	5.386	0.3331	0.4991	1.245×10^{-2}	$\sim\pi$
(31)	$\cos(\frac{5\pi}{14})$	1.132	0.6697	5.161	0.3334	0.5000	1.968×10^{-3}	$\sim\pi$
(32)	$\cos(\frac{2\pi}{7})$	0.8235	0.4557	5.386	0.3331	0.5009	1.245×10^{-2}	~0

Table 5: Numerical results in the case of TBM. We assume that the bound on $\sin^2(\theta_{13})$ is 0.1 and 10% errors for the other two sine squares. The values of δ have a numerical precision of $\mathcal{O}(10^{-6})$. Note that in case of the (11) element being $\cos(\frac{3\pi}{14})$ δ can take arbitrary values (for details see text).

$|(V^{11}_{\rm mix})_{23}|^2$ and $|(V^{11}_{\rm mix})_{33}|^2$ found in appendix A leads to

$$\tan(2\,\theta_l)=\frac{\sin^2(\theta_\nu)-\cos^2\left((\phi_l-\phi_\nu)\,\frac{\rm j}{2}\right)\cos^2(\theta_\nu)}{\cos\left((\phi_l-\phi_\nu)\,\frac{\rm j}{2}+\alpha\right)\,\cos\left((\phi_l-\phi_\nu)\,\frac{\rm j}{2}\right)\,\sin(2\,\theta_\nu)} \tag{7.6}$$

with θ_ν determined by $\sin^2(\theta_{12,13})$. Allowing $\alpha\in[0,2\,\pi)$ one finds the maximal range of θ_l to be $z\leq\theta_l\leq\frac{\pi}{2}-z$ with $z\approx0.4396$ for $\theta_\nu\approx1.139$ and $(\phi_l-\phi_\nu)\,\frac{\rm j}{2}=\frac{3\pi}{14}$ which corresponds to the numerical values given in table 5. Furthermore, eq. (7.6) shows that θ_l is a function of α.
Demanding $\sin^2(\theta_{13})\leq0.025$ removes the possibility that the (11) element of $V_{\rm MNS}$ is determined by group theory, while it leads to expected slight changes in the results of the fits for the rest of the cases.

8. Summary and conclusions

It has been pointed out in [1, 2] that it is possible to predict $|V_{us}|$ as $\cos(\frac{3\pi}{7})\approx0.2225$ with the help of a dihedral symmetry, broken in a non-trivial way. Here we first studied which of the other elements of $V_{\rm CKM}$ can also be described in this way for certain values of the group index n of the dihedral symmetry. For the smallest two appropriate values of n, $n=7$ and $n=14$, this is possible for all elements of the $1-2$ sub-block of $V_{\rm CKM}$. Thereby, the other elements can be fitted by choosing the free angles θ_u and θ_d and the phase α properly. We presented a low energy model for the quark sector with the flavor symmetry D_7. It is broken only spontaneously at the electroweak scale by Higgs fields transforming as doublets under $SU(2)_L$. With a numerical fit we showed that all quark masses and mixing parameters can be accommodated well at the same time. As the VEV configuration determines the subgroup to which the flavor symmetry is broken, it is necessary to investigate whether this can be achieved by the Higgs potential. A detailed study revealed that this is possible. However, there are two obstacles: the Higgs masses turn out to be too small (some of them are even below the LEP bound [17]), if we do not assume additional ingredients such as soft breaking terms in the potential, and secondly, we are only able to accommodate the

VEV configuration as one possible solution of the Higgs potential, but not as a favored one. Moreover, it is well-known that in multi-Higgs doublet models there is in general no mechanism to stabilize a certain VEV configuration. Therefore this model is meant as a proof of principle rather than a realistic model. A way to circumvent these problems is to disentangle the scales of the electroweak and the flavor symmetry breaking by using flavored gauge singlets instead of Higgs doublets and thereby break the dihedral symmetry at higher energies [6]. Accounting for the fact that the Cabibbo angle θ_C is roughly an order of magnitude larger than the two other mixing angles θ_{13}^q and θ_{23}^q one can look for models in which θ_C is given in terms of group theoretical quantities and θ_{13}^q and θ_{23}^q vanish at LO. For this purpose, we can either simply reduce the number of Higgs fields in the model by omitting some fields which are allowed to have a non-trivial VEV in principle or we can break the dihedral symmetry down to one of its dihedral subgroups, D_q, $q > 1$, instead of Z_2. However, for the second possibility we have to use D_{14} instead of D_7. The preserved subgroups are then of the form $D_2 =< \mathrm{A}^7, \mathrm{B}\,\mathrm{A}^m >$. Also here two different D_2 groups are preserved in the up quark and down quark sector in order to generate a non-vanishing Cabibbo angle. One possible choice is $m_u = 6$ and $m_d = 0$. Finally, we also studied the lepton mixing matrix V_{MNS} numerically under the assumption that neutrinos are Dirac particles and normally ordered. Since the elements of V_{MNS} are much less constrained than the ones of V_{CKM} much more combinations of the group theoretical quantities n, j, m_l and m_ν can be used in order to describe an element of V_{MNS}. However, since we expect that the leptons transform under the same flavor symmetry as the quarks, we only considered the cases $n = 7$ and $n = 14$. A numerical analysis shows that the experimental fit values of the mixing angles can be accommodated well in most of the cases. A common feature of all fits is the fact that J_{CP} vanishes. We also studied how well one could mimic the TBM scenario. This is possible in various cases. The case, in which the (11) element of $|V_{\mathrm{MNS}}|$ is determined by group theory, is thereby the most interesting one, since only this case allows for non-trivial CP violation. However, the value of $\sin^2(\theta_{13})$ turns out to be very large.

We focussed on the case of Dirac neutrinos, since then all formulae found in case of the quarks are applicable also to the lepton sector. But, neutrinos can be Majorana particles as well. If we assume that they acquire masses from Higgs triplets only, the analysis done in section 7 is not changed. Things can change, if we consider the type 1 seesaw instead, since we then deal with the Dirac neutrino and the right-handed Majorana mass matrices, which can preserve different subgroups of the flavor symmetry.[7] Beyond that, we could encounter new results with the neutrino mass hierarchy being inverted ($m_3 < m_1 < m_2$).

Our study is by no means a complete study of all possible mixing structures which can in principle arise from a dihedral flavor symmetry with residual subgroups. For example, in all cases we presented here the subgroups, preserved in the up and down quark sector, have the same group structure (either Z_2 or D_2). In general, however, these group structures could be different, as employed in [2, 4−6, 18, 19].

Finally, let us remark that a common feature of the model(s) shown here is the need for an additional $Z_n{}^{(aux)}$ symmetry which can separate the different sectors according to the

[7]This is, for example, the case in the models [18, 19] by Grimus and Lavoura.

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different conserved subgroups of the flavor symmetry. Due to such an additional symmetry an embedding of these models into an SO(10) GUT is in general not straightforward. However, assigning the quarks to

$$Q_1, u_1^c \sim (\underline{\mathbf{1}}_{\mathbf{1}}, +1)\ ,\ \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}, \begin{pmatrix} u_2^c \\ u_3^c \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, +1)\ ,\ d_1^c \sim (\underline{\mathbf{1}}_{\mathbf{1}}, -1)\ ,\ \begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix} \sim (\underline{\mathbf{2}}_{\mathbf{1}}, -1) \tag{8.1}$$

under $D_7 \times Z_2^{(aux)}$ as done in section 4.1.2 still allows an embedding into SU(5) multiplets.

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A. Possible forms of $V_{\rm mix}$

According to the three possible identifications of the eigenvalue $c - d$ there exist three possible diagonalization matrices in each sector (up and down sector, charged lepton and neutrino sector) U, U' and U'' which are shown in section 3.1. Out of these one can form nine possible mixing matrices $V_{\rm mix}^{ab} = W_1^T W_2^\star$ with $a, b = 1, 2, 3$ and $W_i \in \{U, U', U''\}$ where W_i depends on the group theoretical phase ϕ_i (the index m_i) and contains the parameters θ_i and β_i. As shown above, $V_{\rm mix}^{ab}$ all have the property that the element (ab) is completely determined by group theory. In the following we abbreviate $\beta_1 - \beta_2$ with α, $\sin(\theta_i)$ with s_i and $\cos(\theta_i)$ with c_i.

$$V_{\rm mix}^{11} = \frac{1}{2}\begin{pmatrix} 1+e^{i(\phi_1-\phi_2)\mathrm{j}} & (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 \\ -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 \\ (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 \end{pmatrix}$$

$$V_{\rm mix}^{12} = \frac{1}{2}\begin{pmatrix} (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & 1+e^{i(\phi_1-\phi_2)\mathrm{j}} & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 \\ (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 \\ -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 \end{pmatrix}$$

$$V_{\rm mix}^{13} = \frac{1}{2}\begin{pmatrix} (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 & 1+e^{i(\phi_1-\phi_2)\mathrm{j}} \\ (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 & -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 \\ -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 & (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 \end{pmatrix}$$

$$V_{\rm mix}^{21} = \frac{1}{2}\begin{pmatrix} -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 \\ 1+e^{i(\phi_1-\phi_2)\mathrm{j}} & (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 \\ (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 \end{pmatrix}$$

$$V_{\rm mix}^{22} = \frac{1}{2}\begin{pmatrix} (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 \\ (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & 1+e^{i(\phi_1-\phi_2)\mathrm{j}} & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 \\ -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 \end{pmatrix}$$

$$V_{\rm mix}^{23} = \frac{1}{2}\begin{pmatrix} (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 & -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 \\ (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 & 1+e^{i(\phi_1-\phi_2)\mathrm{j}} \\ -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 & (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 \end{pmatrix}$$

$$V_{\rm mix}^{31} = \frac{1}{2}\begin{pmatrix} -(e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})s_1 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})s_1c_2+2e^{i\alpha}c_1s_2 \\ (e^{-i\phi_1\mathrm{j}}-e^{-i\phi_2\mathrm{j}})c_1 & -(1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)\mathrm{j}})c_1c_2+2e^{i\alpha}s_1s_2 \\ 1+e^{i(\phi_1-\phi_2)\mathrm{j}} & (e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})s_2 & -(e^{i\phi_1\mathrm{j}}-e^{i\phi_2\mathrm{j}})c_2 \end{pmatrix}$$

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$$V^{32}_{\rm mix}=\frac{1}{2}\begin{pmatrix} (1+e^{-i(\phi_1-\phi_2)j})s_1 s_2+2e^{i\alpha}c_1c_2 & -(e^{-i\phi_1 j}-e^{-i\phi_2 j})s_1 & -(1+e^{-i(\phi_1-\phi_2)j})s_1c_2+2e^{i\alpha}c_1s_2 \\ -(1+e^{-i(\phi_1-\phi_2)j})c_1s_2+2e^{i\alpha}s_1c_2 & (e^{-i\phi_1 j}-e^{-i\phi_2 j})c_1 & (1+e^{-i(\phi_1-\phi_2)j})c_1c_2+2e^{i\alpha}s_1s_2 \\ (e^{i\phi_1 j}-e^{i\phi_2 j})s_2 & 1+e^{i(\phi_1-\phi_2)j} & -(e^{i\phi_1 j}-e^{i\phi_2 j})c_2 \end{pmatrix}$$

$$V^{33}_{\rm mix}=\frac{1}{2}\begin{pmatrix} (1+e^{-i(\phi_1-\phi_2)j})s_1 s_2+2e^{i\alpha}c_1c_2 & -(1+e^{-i(\phi_1-\phi_2)j})s_1c_2+2e^{i\alpha}c_1s_2 & -(e^{-i\phi_1 j}-e^{-i\phi_2 j})s_1 \\ -(1+e^{-i(\phi_1-\phi_2)j})c_1s_2+2e^{i\alpha}s_1c_2 & (1+e^{-i(\phi_1-\phi_2)j})c_1c_2+2e^{i\alpha}s_1s_2 & (e^{-i\phi_1 j}-e^{-i\phi_2 j})c_1 \\ (e^{i\phi_1 j}-e^{i\phi_2 j})s_2 & -(e^{i\phi_1 j}-e^{i\phi_2 j})c_2 & 1+e^{i(\phi_1-\phi_2)j} \end{pmatrix}$$

The measure of CP violation $J^{ab}_{\rm CP}$ is given for the matrices $V^{ab}_{\rm mix}$ as

$$J^{11}_{\rm CP}=J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,\qquad J^{12}_{\rm CP}=-J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,$$
$$J^{13}_{\rm CP}=J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha) \tag{A.1}$$
$$J^{21}_{\rm CP}=-J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,\qquad J^{22}_{\rm CP}=J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,$$
$$J^{23}_{\rm CP}=-J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha) \tag{A.2}$$
$$J^{31}_{\rm CP}=J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,\qquad J^{32}_{\rm CP}=-J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)\,,$$
$$J^{33}_{\rm CP}=J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha) \tag{A.3}$$

$$\text{with}\quad J_{\rm CP}(j,\phi_1,\phi_2;\theta_1,\theta_2,\alpha)=-\frac{1}{8}\sin((\phi_1-\phi_2)\,j)\,\sin\left(\frac{1}{2}(\phi_1-\phi_2)\,j\right)$$
$$\times\sin(2\,\theta_1)\,\sin(2\,\theta_2)\,\sin\left(\frac{1}{2}(\phi_1-\phi_2)\,j+\alpha\right) \tag{A.4}$$

B. Group theory of D_7

The group D_7 has two one- and three two-dimensional irreducible representations which we denote as $\underline{\mathbf{1}}_\mathbf{1}$, $\underline{\mathbf{1}}_\mathbf{2}$, $\underline{\mathbf{2}}_\mathbf{1}$, $\underline{\mathbf{2}}_\mathbf{2}$ and $\underline{\mathbf{2}}_\mathbf{3}$. $\underline{\mathbf{1}}_\mathbf{1}$ is the trivial representation of the group. All two-dimensional representations are faithful. The order of the group is 14. The generator relations for the two generators A and B are:

$$\mathrm{A}^7=1\,,\quad \mathrm{B}^2=1\,,\quad \mathrm{ABA}=\mathrm{B}\,.$$

A and B can be chosen to be

$$\mathrm{A}=\begin{pmatrix} e^{\frac{2\pi i}{7}} & 0 \\ 0 & e^{-\frac{2\pi i}{7}}\end{pmatrix}\,,\quad \mathrm{B}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \text{for}\quad \underline{\mathbf{2}}_\mathbf{1}$$
$$\mathrm{A}=\begin{pmatrix} e^{\frac{4\pi i}{7}} & 0 \\ 0 & e^{-\frac{4\pi i}{7}}\end{pmatrix}\,,\quad \mathrm{B}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \text{for}\quad \underline{\mathbf{2}}_\mathbf{2}$$
$$\mathrm{A}=\begin{pmatrix} e^{\frac{6\pi i}{7}} & 0 \\ 0 & e^{-\frac{6\pi i}{7}}\end{pmatrix}\,,\quad \mathrm{B}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \text{for}\quad \underline{\mathbf{2}}_\mathbf{3}$$

For the one-dimensional representations $\underline{\mathbf{1}}_\mathbf{1}$ and $\underline{\mathbf{1}}_\mathbf{2}$ A and B can be found in the character table table 6. The Kronecker products are:

$$\underline{\mathbf{1}}_\mathbf{1}\times\mu=\mu\,,\qquad \underline{\mathbf{1}}_\mathbf{2}\times\underline{\mathbf{1}}_\mathbf{2}=\underline{\mathbf{1}}_\mathbf{1}\,,\qquad \underline{\mathbf{1}}_\mathbf{2}\times\underline{\mathbf{2}}_\mathbf{i}=\underline{\mathbf{2}}_\mathbf{i}$$
$$[\underline{\mathbf{2}}_\mathbf{1}\times\underline{\mathbf{2}}_\mathbf{1}]=\underline{\mathbf{1}}_\mathbf{1}+\underline{\mathbf{2}}_\mathbf{2}\,,\qquad \{\underline{\mathbf{2}}_\mathbf{1}\times\underline{\mathbf{2}}_\mathbf{1}\}=\underline{\mathbf{1}}_\mathbf{2}$$
$$[\underline{\mathbf{2}}_\mathbf{2}\times\underline{\mathbf{2}}_\mathbf{2}]=\underline{\mathbf{1}}_\mathbf{1}+\underline{\mathbf{2}}_\mathbf{3}\,,\qquad \{\underline{\mathbf{2}}_\mathbf{2}\times\underline{\mathbf{2}}_\mathbf{2}\}=\underline{\mathbf{1}}_\mathbf{2}$$

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	classes				
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
G	$\mathbb{1}$	A	A^2	A^3	B
$^\circ\mathcal{C}_i$	1	2	2	2	7
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	7	7	7	2
$\underline{\mathbf{1_1}}$	1	1	1	1	1
$\underline{\mathbf{1_2}}$	1	1	1	1	-1
$\underline{\mathbf{2_1}}$	2	$2\cos(\varphi)$	$2\cos(2\varphi)$	$2\cos(3\varphi)$	0
$\underline{\mathbf{2_2}}$	2	$2\cos(2\varphi)$	$2\cos(4\varphi)$	$2\cos(6\varphi)$	0
$\underline{\mathbf{2_3}}$	2	$2\cos(3\varphi)$	$2\cos(6\varphi)$	$2\cos(9\varphi)$	0

Table 6: Character table of the group D_7. φ is $\frac{2\pi}{7}$. $\mathcal{C}_i$ are the classes of the group, $^\circ\mathcal{C}_i$ is the order of the i^{th} class, i.e. the number of distinct elements contained in this class, $^\circ\mathrm{h}_{\mathcal{C}_i}$ is the order of the elements S in the class $\mathcal{C}_i$, i.e. the smallest integer (> 0) for which the equation $S^{\,^\circ\mathrm{h}_{\mathcal{C}_i}} = \mathbb{1}$ holds. Furthermore the table contains one representative for each class $\mathcal{C}_i$ given as product of the generators A and B of the group.

$$[\underline{\mathbf{2_3}} \times \underline{\mathbf{2_3}}] = \underline{\mathbf{1_1}} + \underline{\mathbf{2_1}}\ , \qquad \{\underline{\mathbf{2_3}} \times \underline{\mathbf{2_3}}\} = \underline{\mathbf{1_2}}$$

$$\underline{\mathbf{2_1}} \times \underline{\mathbf{2_2}} = \underline{\mathbf{2_1}} + \underline{\mathbf{2_3}}\ , \qquad \underline{\mathbf{2_1}} \times \underline{\mathbf{2_3}} = \underline{\mathbf{2_2}} + \underline{\mathbf{2_3}}\ , \qquad \underline{\mathbf{2_2}} \times \underline{\mathbf{2_3}} = \underline{\mathbf{2_1}} + \underline{\mathbf{2_2}}\ ,$$

where μ is any representation of the group and $[\nu \times \nu]$ denotes the symmetric part of the product $\nu \times \nu$, while $\{\nu \times \nu\}$ is the anti-symmetric one.
The Clebsch Gordan coefficients are trivial for $\underline{\mathbf{1_1}} \times \mu$ and $\underline{\mathbf{1_2}} \times \underline{\mathbf{1_2}}$. For $\underline{\mathbf{1_2}} \times \underline{\mathbf{2_i}}$ a non-trivial sign appears

$$\begin{pmatrix} B\, a_1 \\ -B\, a_2 \end{pmatrix} \sim \underline{\mathbf{2_i}}$$

for $B \sim \underline{\mathbf{1_2}}$ and $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2_i}}$. $\underline{\mathbf{1_1}}$ and $\underline{\mathbf{1_2}}$ of $\underline{\mathbf{2_i}} \times \underline{\mathbf{2_i}}$ are of the form

$$a_1\, a_2' + a_2\, a_1' \sim \underline{\mathbf{1_1}}\ , \quad a_1\, a_2' - a_2\, a_1' \sim \underline{\mathbf{1_2}}$$

for $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \sim \underline{\mathbf{2_i}}$. The two-dimensional representations also contained in these products read:

$$\text{for i} = 1: \begin{pmatrix} a_1\, a_1' \\ a_2\, a_2' \end{pmatrix} \sim \underline{\mathbf{2_2}}\ , \quad \text{for i} = 2: \begin{pmatrix} a_2\, a_2' \\ a_1\, a_1' \end{pmatrix} \sim \underline{\mathbf{2_3}}\ , \quad \text{for i} = 3: \begin{pmatrix} a_2\, a_2' \\ a_1\, a_1' \end{pmatrix} \sim \underline{\mathbf{2_1}}\ .$$

For the rest of the products $\underline{\mathbf{2_i}} \times \underline{\mathbf{2_j}}$ we get:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2_1}}\ , \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{\mathbf{2_2}}\ : \qquad \begin{pmatrix} a_2\, b_1 \\ a_1\, b_2 \end{pmatrix} \sim \underline{\mathbf{2_1}}\ , \qquad \begin{pmatrix} a_1\, b_1 \\ a_2\, b_2 \end{pmatrix} \sim \underline{\mathbf{2_3}}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \,, \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \;: \qquad \begin{pmatrix} a_2\, b_1 \\ a_1\, b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \,, \qquad \begin{pmatrix} a_2\, b_2 \\ a_1\, b_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \,, \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \;: \qquad \begin{pmatrix} a_2\, b_1 \\ a_1\, b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \,, \qquad \begin{pmatrix} a_2\, b_2 \\ a_1\, b_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}$$

All these formulae are just special cases of the more general formulae given in [13, 1] which hold for dihedral groups D_n with an arbitrary index n.

C. Higgs potential

The potential V_u of H^u_s and $H^u_{1,2}$ is of the same form as V_3 in eq. (5.1) with all parameters carrying an additional upper index u. As already stated, the potential contains an accidental U(1) symmetry. The most general potential involving only the scalar fields H^d_s, $H^d_{1,2}$ and $\chi^d_{1,2}$ is

$$\begin{aligned}
V_d = & -(\mu^d_s)^2 H^{d\dagger}_s H^d_s - (\mu^d_D)^2 \left(\sum_{i=1}^{2} H^{d\dagger}_i H^d_i\right) - (\tilde{\mu}^d_D)^2 \left(\sum_{i=1}^{2} \chi^{d\dagger}_i \chi^d_i\right) \\
& + \lambda^d_s (H^{d\dagger}_s H^d_s)^2 + \lambda^d_1 \left(\sum_{i=1}^{2} H^{d\dagger}_i H^d_i\right)^2 + \tilde{\lambda}^d_1 \left(\sum_{i=1}^{2} \chi^{d\dagger}_i \chi^d_i\right)^2 + \lambda^d_2 (H^{d\dagger}_1 H^d_1 - H^{d\dagger}_2 H^d_2)^2 \\
& + \tilde{\lambda}^d_2 (\chi^{d\dagger}_1 \chi^d_1 - \chi^{d\dagger}_2 \chi^d_2)^2 + \lambda^d_3 |H^{d\dagger}_1 H^d_2|^2 + \tilde{\lambda}^d_3 |\chi^{d\dagger}_1 \chi^d_2|^2 + \sigma^d_1 (H^{d\dagger}_s H^d_s) \left(\sum_{i=1}^{2} H^{d\dagger}_i H^d_i\right) \\
& + \tilde{\sigma}^d_1 (H^{d\dagger}_s H^d_s) \left(\sum_{i=1}^{2} \chi^{d\dagger}_i \chi^d_i\right) + \{\sigma^d_2 (H^{d\dagger}_s H^d_1)(H^{d\dagger}_s H^d_2) + \text{h.c.}\} + \{\tilde{\sigma}^d_2 (H^{d\dagger}_s \chi^d_1)(H^{d\dagger}_s \chi^d_2) + \text{h.c.}\} \\
& + \sigma^d_3 \left(\sum_{i=1}^{2} |H^{d\dagger}_s H^d_i|^2\right) + \tilde{\sigma}^d_3 \left(\sum_{i=1}^{2} |H^{d\dagger}_s \chi^d_i|^2\right) + \tau^d_1 \left(\sum_{i=1}^{2} H^{d\dagger}_i H^d_i\right) \left(\sum_{i=1}^{2} \chi^{d\dagger}_i \chi^d_i\right) \\
& + \tau^d_2 (H^{d\dagger}_1 H^d_1 - H^{d\dagger}_2 H^d_2)(\chi^{d\dagger}_1 \chi^d_1 - \chi^{d\dagger}_2 \chi^d_2) + \{\tau^d_3 (H^{d\dagger}_1 \chi^d_1)(H^{d\dagger}_2 \chi^d_2) + \text{h.c.}\} \\
& + \tau^d_4 \left(\sum_{i=1}^{2} |H^{d\dagger}_i \chi^d_i|^2\right) + \{\tau^d_5 (H^{d\dagger}_1 \chi^d_2)(H^{d\dagger}_2 \chi^d_1) + \text{h.c.}\} + \tau^d_6 (\,|H^{d\dagger}_1 \chi^d_2|^2 + |H^{d\dagger}_2 \chi^d_1|^2) \\
& + \{\tau^d_7 \{(H^{d\dagger}_2 \chi^d_1)(\chi^{d\dagger}_2 \chi^d_1) + (H^{d\dagger}_1 \chi^d_2)(\chi^{d\dagger}_1 \chi^d_2)\} + \text{h.c.}\} \\
& + \{\omega^d_1 \{(H^{d\dagger}_s H^d_1)(H^{d\dagger}_2 \chi^d_2) + (H^{d\dagger}_s H^d_2)(H^{d\dagger}_1 \chi^d_1)\} + \text{h.c.}\} \\
& + \{\omega^d_2 \{(H^{d\dagger}_s H^d_1)(\chi^{d\dagger}_1 H^d_1) + (H^{d\dagger}_s H^d_2)(\chi^{d\dagger}_2 H^d_2)\} + \text{h.c.}\} \\
& + \{\omega^d_3 \{(H^{d\dagger}_s \chi^d_1)(H^{d\dagger}_1 H^d_2) + (H^{d\dagger}_s \chi^d_2)(H^{d\dagger}_2 H^d_1)\} + \text{h.c.}\}
\end{aligned} \tag{C.1}$$

This five Higgs potential is free from accidental symmetries. However, the combined potential $V_u + V_d$ has an accidental $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ symmetry. It is broken explicitly by mixing terms, which couple the Higgs fields $H^u_{s,1,2}$ and $H^d_{s,1,2}/\chi^d_{1,2}$. V_{mixed} contains all

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such terms, which are invariant under the symmetry $D_7 \times Z_2^{(aux)}$:

$$
\begin{aligned}
V_{\rm mixed} =\ & \kappa_1 (H_s^{u\dagger} H_s^u)(H_s^{d\dagger} H_s^d) + \{\kappa_2 (H_s^{u\dagger} H_s^d)^2 + \text{h.c.}\} + \kappa_3 |H_s^{u\dagger} H_s^d|^2 \qquad \text{(C.2)} \\
& + \kappa_4 \left(\sum_{i=1}^{2} H_i^{u\dagger} H_i^u\right)\left(\sum_{i=1}^{2} H_i^{d\dagger} H_i^d\right) + \tilde{\kappa}_4 \left(\sum_{i=1}^{2} H_i^{u\dagger} H_i^u\right)\left(\sum_{i=1}^{2} \chi_i^{d\dagger} \chi_i^d\right) \\
& + \{\kappa_5 \left(\sum_{i=1}^{2} H_i^{u\dagger} H_i^d\right)^2 + \text{h.c.}\} + \kappa_6 |H_1^{u\dagger} H_1^d + H_2^{u\dagger} H_2^d|^2 \\
& + \kappa_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u)(H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d) + \tilde{\kappa}_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u)(\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d) \\
& + \{\kappa_8 (H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d)^2 + \text{h.c.}\} + \{\tilde{\kappa}_{[5-8]} (H_1^{u\dagger} \chi_1^d)(H_2^{u\dagger} \chi_2^d) + \text{h.c.}\} \\
& + \kappa_9 |H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d|^2 + \tilde{\kappa}_{[6+9]} (|H_1^{u\dagger} \chi_1^d|^2 + |H_2^{u\dagger} \chi_2^d|^2) \\
& + \kappa_{10} \{(H_2^{u\dagger} H_1^u)(H_1^{d\dagger} H_2^d) + \text{h.c.}\} + \{\kappa_{11} (H_2^{u\dagger} H_1^d)(H_1^{u\dagger} H_2^d) + \text{h.c.}\} \\
& + \{\tilde{\kappa}_{11} (H_1^{u\dagger} \chi_2^d)(H_2^{u\dagger} \chi_1^d) + \text{h.c.}\} + \kappa_{12} (|H_2^{u\dagger} H_1^d|^2 + |H_1^{u\dagger} H_2^d|^2) \\
& + \tilde{\kappa}_{12} (|H_1^{u\dagger} \chi_2^d|^2 + |H_2^{u\dagger} \chi_1^d|^2) + \kappa_{13} (H_s^{u\dagger} H_s^u)\left(\sum_{i=1}^{2} H_i^{d\dagger} H_i^d\right) + \tilde{\kappa}_{13} (H_s^{u\dagger} H_s^u)\left(\sum_{i=1}^{2} \chi_i^{d\dagger} \chi_i^d\right) \\
& + \{\kappa_{14} (H_s^{u\dagger} H_1^d)(H_s^{u\dagger} H_2^d) + \text{h.c.}\} + \{\tilde{\kappa}_{14} (H_s^{u\dagger} \chi_1^d)(H_s^{u\dagger} \chi_2^d) + \text{h.c.}\} \\
& + \kappa_{15} (|H_s^{u\dagger} H_1^d|^2 + |H_s^{u\dagger} H_2^d|^2) + \tilde{\kappa}_{15} (|H_s^{u\dagger} \chi_1^d|^2 + |H_s^{u\dagger} \chi_2^d|^2) \\
& + \kappa_{16} (H_s^{d\dagger} H_s^d)\left(\sum_{i=1}^{2} H_i^{u\dagger} H_i^u\right) + \{\kappa_{17} (H_s^{d\dagger} H_1^u)(H_s^{d\dagger} H_2^u) + \text{h.c.}\} + \kappa_{18}\left(\sum_{i=1}^{2} |H_s^{d\dagger} H_i^u|^2\right) \\
& + \left\{\kappa_{19} (H_s^{u\dagger} H_s^d)\left(\sum_{i=1}^{2} H_i^{u\dagger} H_i^d\right) + \text{h.c.}\right\} + \left\{\kappa_{20} (H_s^{u\dagger} H_s^d)\left(\sum_{i=1}^{2} H_i^{d\dagger} H_i^u\right) + \text{h.c.}\right\} \\
& + \{\kappa_{21} \{(H_s^{u\dagger} H_1^u)(H_s^{d\dagger} H_2^d) + (H_s^{u\dagger} H_2^u)(H_s^{d\dagger} H_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{22} \{(H_s^{u\dagger} H_1^d)(H_s^{d\dagger} H_2^u) + (H_s^{u\dagger} H_2^d)(H_s^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{23} \{(H_s^{u\dagger} H_1^u)(H_1^{d\dagger} H_s^d) + (H_s^{u\dagger} H_2^u)(H_2^{d\dagger} H_s^d)\} + \text{h.c.}\} \\
& + \{\kappa_{24} \{(H_s^{u\dagger} H_1^d)(H_1^{u\dagger} H_s^d) + (H_s^{u\dagger} H_2^d)(H_2^{u\dagger} H_s^d)\} + \text{h.c.}\} \\
& + \{\kappa_{25} \{(H_s^{d\dagger} H_1^u)(H_2^{u\dagger} \chi_2^d) + (H_s^{d\dagger} H_2^u)(H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{26} \{(H_s^{d\dagger} H_1^u)(\chi_1^{d\dagger} H_1^u) + (H_s^{d\dagger} H_2^u)(\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} \\
& + \{\kappa_{27} \{(H_s^{d\dagger} \chi_1^d)(H_1^{u\dagger} H_2^u) + (H_s^{d\dagger} \chi_2^d)(H_2^{u\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{28} \{(H_s^{u\dagger} H_1^d)(H_2^{u\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^d)(H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{29} \{(H_s^{u\dagger} H_1^d)(\chi_1^{d\dagger} H_1^u) + (H_s^{u\dagger} H_2^d)(\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} \\
& + \{\kappa_{30} \{(H_s^{u\dagger} \chi_1^d)(H_1^{d\dagger} H_2^u) + (H_s^{u\dagger} \chi_2^d)(H_2^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{31} \{(H_s^{u\dagger} \chi_1^d)(H_1^{u\dagger} H_2^d) + (H_s^{u\dagger} \chi_2^d)(H_2^{u\dagger} H_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{32} \{(H_s^{u\dagger} H_1^u)(H_2^{d\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^u)(H_1^{d\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{33} \{(H_s^{u\dagger} H_1^u)(\chi_1^{d\dagger} H_1^d) + (H_s^{u\dagger} H_2^u)(\chi_2^{d\dagger} H_2^d)\} + \text{h.c.}\}
\end{aligned}
$$

In our numerical analysis we restricted ourselves to the inclusion of a minimal number of terms from $V_{\rm mixed}$ which break all accidental symmetries such that only three Higgs

particles remain massless which are eaten by the $W^\pm$ and Z^0 boson. As explained in the main part of the text, the three terms κ_2, κ_5 and κ_{19} are sufficient.
The numerical example in section 4.1.1 and section 4.2.1 needs the following VEV configuration

$$\langle H_s^{d,u}\rangle{=}61.5\,\text{GeV}, \qquad \langle H_1^d\rangle{=}\langle H_2^d\rangle{=}\langle \chi_1^d\rangle{=}\langle \chi_2^d\rangle{=}61.5\,\text{GeV}, \quad \langle H_1^u\rangle{=}61.5\,\text{e}^{-\frac{3\pi i}{7}}\,\text{GeV}$$
$$\text{and } \langle H_2^u\rangle{=}61.5\,\text{e}^{\frac{3\pi i}{7}}\,\text{GeV}$$

which allows real parameters in the potential V_d, as all fields H_s^d, $H_{1,2}^d$ and $\chi_{1,2}^d$ have real VEVs. Furthermore we can remove the phase of σ_2^u such that we are left with three complex parameters stemming from V_{mixed}.
The mass parameters are chosen to be around the electroweak scale, i.e. $\mu_s^u = 100\,\text{GeV}$, $\mu_D^u = 200\,\text{GeV}$, $\mu_s^d = 100\,\text{GeV}$, $\mu_D^d = 200\,\text{GeV}$ and $\tilde{\mu}_D^d = 150\,\text{GeV}$. One possible setup of quartic couplings is then:

$$\begin{array}{lllll}
\lambda_s^u{=}0.959337\,, & \lambda_1^u{=}2.52548\,, & \lambda_2^u{=}0.374967\,, & \lambda_3^u{=}-0.588842\,, & \sigma_1^u{=}1.62353\,,\\
\sigma_2^u{=}-0.276964\,, & \sigma_3^u{=}-0.283914\,, & & & \\
\lambda_s^d{=}1.70438\,, & \lambda_1^d{=}3.76598\,, & \tilde{\lambda}_1^d{=}1.47549\,, & \lambda_2^d{=}-0.344036\,, & \tilde{\lambda}_2^d{=}-0.185157\,,\\
\lambda_3^d{=}-0.304589\,, & \tilde{\lambda}_3^d{=}-0.733236\,, & \sigma_1^d{=}0.22429\,, & \tilde{\sigma}_1^d{=}4.6792\,, & \sigma_2^d{=}-0.87457\,,\\
\tilde{\sigma}_2^d{=}-2.0284\,, & \sigma_3^d{=}0.961454\,, & \tilde{\sigma}_3^d{=}0.649984\,, & \tau_1^d{=}2.96557\,, & \tau_2^d{=}1.22903\,,\\
\tau_3^d{=}-2.02133\,, & \tau_4^d{=}-1.22242\,, & \tau_5^d{=}-2.31577\,, & \tau_6^d{=}2.38236\,, & \tau_7^d{=}-0.660102\,,\\
\omega_1^d{=}0.452165\,, & \omega_2^d{=}-2.112\,, & \omega_3^d{=}-1.63452\,, & &
\end{array}$$
$$\kappa_2{=}-0.638073+i\,0.0277608\,, \quad \kappa_5{=}0.312782+i\,0.140162\,, \quad \kappa_{19} = -0.278402-i\,0.124756$$

Note that all parameters have absolute values smaller than 5 and hence they are still in the perturbative regime. With these parameter values we obtain the desired VEV structure. The Higgs masses are then 513 GeV, 499 GeV, 426 GeV, 414 GeV, 386 GeV, 365 GeV, 321 GeV, 266 GeV, 246 GeV, 227 GeV, 178 GeV, 159 GeV, 134 GeV, 81 GeV and 55 GeV for the neutral scalars. Due to the explicit CP violation in the potential we can no longer distinguish between scalars and pseudo-scalars. For the charged scalar fields we get 367 GeV, 333 GeV, 294 GeV, 261 GeV, 145 GeV, 115 GeV and 55 GeV. They are therefore in general too light to pass the constraints coming from direct searches as well as from bounds on FCNCs. Nevertheless, soft breaking terms of mass dimension two of the order of 10 TeV could lift the masses above these experimental bounds.

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